

# ON THE SUM OF A SOBOLEV SPACE AND A WEIGHTED $L_p$ -SPACE

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## Abstract

Let  $p > n$  and let  $L_p^1(\mathbf{R}^n)$  be a homogeneous Sobolev space. For an arbitrary Borel measure  $\mu$  on  $\mathbf{R}^n$  we give a constructive characterization of the space

$$\Sigma = L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu).$$

We express the norm in this space in terms of certain oscillations with respect to the measure  $\mu$ . This enables us to describe the  $K$ -functional for the couple  $(L_p(\mathbf{R}^n; \mu), L_p^1(\mathbf{R}^n))$  in terms of these oscillations, and to prove that this couple is quasi-linearizable.

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## 1. Introduction.

Let  $\mu$  be a non-trivial non-negative Borel measure on  $\mathbf{R}^n$  and let  $L_p(\mathbf{R}^n; \mu)$ ,  $1 \leq p \leq \infty$ , be the space  $L_p$  on  $\mathbf{R}^n$  with respect to the measure  $\mu$ , with the standard norm

$$\|f\|_{L_p(\mathbf{R}^n; \mu)} = \left( \int_{\mathbf{R}^n} |f|^p d\mu \right)^{\frac{1}{p}}.$$

By  $L_p^1(\mathbf{R}^n)$  we denote the homogeneous Sobolev space consisting of all (equivalence classes of) real valued functions  $f \in L_{p,loc}(\mathbf{R}^n)$  whose distributional partial derivatives of the first order belong to the space  $L_p(\mathbf{R}^n)$ . We equip the space  $L_p^1(\mathbf{R}^n)$  with the seminorm

$$\|f\|_{L_p^1(\mathbf{R}^n)} := \|\nabla f\|_{L_p(\mathbf{R}^n)}.$$

This paper is devoted to the following main

**Problem 1.1** *Given a function  $f \in L_{p,loc}(\mathbf{R}^n; \mu)$ , how can we tell whether  $f$  belongs to  $L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$ , i.e., whether there exist functions  $f_1 \in L_p^1(\mathbf{R}^n)$  and  $f_2 \in L_p(\mathbf{R}^n)$  such that  $f = f_1 + f_2$ ?*

We also consider a quantitative version of Problem 1.1 related to calculation of the norm of  $f$  in the space

$$\Sigma := L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu).$$

As usual, the space  $\Sigma$  is normed by

$$\|f\|_{\Sigma} := \inf\{\|f_1\|_{L_p^1(\mathbf{R}^n)} + \|f_2\|_{L_p(\mathbf{R}^n; \mu)} : f_1 + f_2 = f, f_1 \in L_p^1(\mathbf{R}^n), f_2 \in L_p(\mathbf{R}^n; \mu)\}.$$

**Problem 1.2** *What is the order of magnitude of the norm of a function  $f$  in the space  $\Sigma = L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$ ?*

In this paper we solve Problems 1.1 and 1.2 by presenting a constructive formula for calculation of the order of magnitude of the norm in the space  $L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$ . This formula is expressed in terms of certain local oscillations of functions with respect to the measure  $\mu$ .

Before we formulate the main result of the paper we need to define several notions and fix some notation:

Throughout this paper, the word “cube” will mean a closed cube in  $\mathbf{R}^n$  whose sides are parallel to the coordinate axes. We let  $Q(x, r)$  denote the cube in  $\mathbf{R}^n$  centered at  $x$  with side length  $2r$ . Given  $\alpha > 0$  and a cube  $Q$  we let  $\alpha Q$  denote the dilation of  $Q$  with respect to its center by a factor of  $\alpha$ . (Thus  $\alpha Q(x, r) = Q(x, \alpha r)$ .) The Lebesgue measure of a measurable set  $A \subset \mathbf{R}^n$  will be denoted by  $|A|$ .

Here now is the main result of our paper:

**Theorem 1.3** *Let  $n < p < \infty$  and let  $\mu$  be a non-trivial non-negative Borel measure on  $\mathbf{R}^n$ . A function  $f \in L_{p,loc}(\mathbf{R}^n; \mu)$  belongs to the space  $L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$  if and only if there exists a positive constant  $\lambda$  which satisfies the following conditions for a certain absolute positive constant  $\gamma$ :*

*Let  $\mathcal{Q}$  be an arbitrary finite family of pairwise disjoint cubes in  $\mathbf{R}^n$ . Suppose that to each cube  $Q \in \mathcal{Q}$  we have arbitrarily assigned two cubes  $Q', Q'' \in \mathcal{Q}$  such that*

$$(1.1) \quad Q' \cup Q'' \subset \gamma Q.$$

*Then the following inequality*

$$(1.2) \quad \sum_{Q \in \mathcal{Q}} \frac{(\text{diam } Q)^{n-p} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)}{\{(\text{diam } Q')^{n-p} + \mu(Q')\} \{(\text{diam } Q'')^{n-p} + \mu(Q'')\}} \leq \lambda$$

*holds. Furthermore,*

$$\|f\|_{\Sigma} \sim \inf \lambda^{\frac{1}{p}}$$

*with constants of equivalence depending only on  $n$  and  $p$ .*

**Remark 1.4** The topic under consideration can be referred to as the Real Interpolation Method for the Banach couple  $\vec{A} = (L_p(\mathbf{R}^n; \mu), L_p^1(\mathbf{R}^n))$ , or, more specifically, as the calculation of the  $K$ -functional

$$K(t; f : \vec{A}) := \inf \{ \|f_1\|_{L_p(\mathbf{R}^n; \mu)} + t \|f_2\|_{L_p^1(\mathbf{R}^n)} : f_1 + f_2 = f, f_1 \in L_p(\mathbf{R}^n; \mu), f_2 \in L_p^1(\mathbf{R}^n) \}.$$

Here  $t$  is a positive number. (See, e.g. [2].)

Thus  $\|f\|_{\Sigma} = K(1; f : \vec{A})$  and

$$K(t; f : \vec{A}) = t \|f\|_{\Sigma_t} \quad \text{where} \quad \Sigma_t := L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \tfrac{1}{t^p} \mu).$$

We recall the classical result of Peetre [11] (see also [1], p. 339), which states that whenever  $1 \leq p \leq \infty$  and  $\mu$  is Lebesgue measure on  $\mathbf{R}^n$ ,

$$K(t; f : (L_p(\mathbf{R}^n), L_p^1(\mathbf{R}^n))) \sim \omega_1(t, f)_{L_p(\mathbf{R}^n)}$$

with constants depending only on  $p$  and  $n$ . Here

$$\omega_1(t, f)_{L_p(\mathbf{R}^n)} = \sup_{\|h\| \leq t} \left\{ \int_{\mathbf{R}^n} |f(x+h) - f(x)|^p dx \right\}^{\frac{1}{p}}$$

is the modulus of smoothness of  $f$  in  $L_p(\mathbf{R}^n)$ .

This result leads us to a solution of Problem 1.2 for the particular case where  $\mu$  is Lebesgue measure multiplied by an arbitrary positive parameter  $s$ . In this case

$$\|f\|_{\Sigma} \sim s^{\frac{1}{p}} \omega_1 \left( s^{-\frac{1}{p}}, f \right)_{L_p(\mathbf{R}^n)}$$

with constants of equivalence depending only on  $n$ . To the best of our knowledge, this measure  $\mu$  is the only example of a measure for which a constructive criterion for the norm of a function in the sum  $L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$  is known so far.  $\triangleleft$

Our second main result, Theorem 1.5, states that the Banach couple  $(L_p^1(\mathbf{R}^n), L_p(\mathbf{R}^n; \mu))$  is *quasi-linearizable*, see [2]. In other words, for every function  $f \in \Sigma = L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$  the functions  $f_1 \in L_p^1(\mathbf{R}^n)$  and  $f_2 \in L_p(\mathbf{R}^n; \mu)$  of an almost optimal decomposition  $f = f_1 + f_2$  can be chosen to *depend linearly* on  $f$ .

**Theorem 1.5** *Let  $n < p < \infty$  and let  $\mu$  be a non-trivial non-negative Borel measure on  $\mathbf{R}^n$ . There exist continuous linear operators*

$$T_1 : L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu) \rightarrow L_p^1(\mathbf{R}^n) \quad \text{and} \quad T_2 : L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu) \rightarrow L_p(\mathbf{R}^n; \mu)$$

*such that*

$$T_1 + T_2 = Id_{\Sigma}$$

*and*

$$\|T_1\|_{\Sigma \rightarrow L_p^1(\mathbf{R}^n)} + \|T_2\|_{\Sigma \rightarrow L_p(\mathbf{R}^n; \mu)} \leq C.$$

*Here  $C = C(n, p)$  is a constant depending only on  $n$  and  $p$ .*

Let us briefly describe the main ideas of the proof of Theorem 1.3. The necessity part of the proof, which we present in **Section 2**, is based on the classical Sobolev-Poincaré inequality for  $L_p^1(\mathbf{R}^n)$ -functions whenever  $p > n$  and the Hardy-Littlewood maximal theorem.

We prove the sufficiency part of the theorem in two steps. The first step is presented in **Section 3** where we construct a closed subset  $E \subset \mathbf{R}^n$  and a certain family  $\mathcal{K}_E$  of pairwise disjoint “well separated” cubes of  $\mathbf{R}^n$  with centers in  $E$ .

This family of cubes possesses certain measure concentration properties (with respect to the measure  $\mu$ ). In particular,  $\mu(K) \sim (\text{diam } K)^{n-p}$  for every cube  $K \in \mathcal{K}_E$ . We also prove that, on the other hand, if  $Q$  is a cube in  $\mathbf{R}^n$ ,  $\theta > 0$  and  $\text{diam } Q \leq \theta \text{ dist}(Q, E)$ , then  $\mu(Q) \leq C(\text{diam } Q)^{n-p}$  where  $C$  is a constant depending only on  $p$  and  $\theta$ .

In **Section 4**, given a function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  satisfying the sufficiency conditions, we construct the functions  $f_1$  and  $f_2 = f - f_1$  of an almost optimal decomposition of  $f$ . We start by defining a function  $\tilde{f}$  on  $E$  by the formula

$$\tilde{f}(x) := \frac{1}{\mu(K(x))} \int_{K(x)} f d\mu, \quad x \in E,$$

where  $K(x)$  denotes the (unique) cube from  $\mathcal{K}_E$  centered at  $x$ . Then we extend  $\tilde{f}$  from  $E$  to all of  $\mathbf{R}^n$  using *the classical Whitney’s extension method*. This gives us  $f_1$  (and therefore of

course also  $f_2 = f - f_1$ ). Section 4 also includes a proof that the function  $f_1$  satisfies the inequality  $\|f_1\|_{L_p^1(\mathbf{R}^n)} \leq C(n, p) \lambda^{\frac{1}{p}}$ .

In **Section 5** we show that the function  $f_2$  satisfies the inequality  $\|f_2\|_{L_p(\mathbf{R}^n; \mu)} \leq C(n, p) \lambda^{\frac{1}{p}}$ . This and the previous inequality prove the sufficiency part of Theorem 1.3.

Note that *the Whitney extension operator is linear*, so that *the functions  $f_1$  and  $f_2$  depend linearly on  $f$* . This proves Theorem 1.5.

In **Section 6** we prove several refinements of Theorem 1.3. Note that the criterion for the norm in the space  $\Sigma = L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$  given in this theorem describes the structure of  $\Sigma$  and shows which properties of a function  $f$  on  $\mathbf{R}^n$  control its almost optimal decomposition into a sum of a function from  $L_p^1(\mathbf{R}^n)$  and a function from  $L_p(\mathbf{R}^n; \mu)$ . At the same time it is not quite clear how one could check the conditions (1.2) of Theorem 1.3 for a given function  $f$  on  $\mathbf{R}^n$ . In fact, these conditions depend on an infinite number of families  $\mathcal{Q}$  of cubes and all possible choices of cubes  $Q', Q'' \in \mathcal{Q}$  satisfying condition (1.1).

Nevertheless a careful examination of our proof of Theorem 1.3 shows that it constructs *a particular family  $\mathcal{Q}$  of cubes and particular mappings  $Q \mapsto Q'$  and  $Q \mapsto Q''$*  satisfying (1.1) depending only on  $p$  and the measure  $\mu$ , and that it is enough to examine the behavior of  $f$  only on this particular family and these particular mappings.

We express this fact by Theorem 6.1 which refines one part of the criterion of Theorem 1.3.

The next refinement of this result, Theorem 6.11, enables us to express the norm of an arbitrary function  $f \in L_{p,loc}(\mathbf{R}^n; \mu)$  as a linear combination of  $p$ -oscillations of  $f$  over a certain family of subsets in  $\mathbf{R}^n$  with fixed covering multiplicity. Note that the coefficients of this linear combination and the family of subsets depend only on  $n, p$ , and the measure  $\mu$ .

We prove this result in Subsection 6.3. Remark that this rather specific refinement of the main result has important applications to problems of characterizations of restrictions of Sobolev functions to closed subsets of  $\mathbf{R}^n$ . (See a discussion at the end of this section.)

The proof of Theorem 6.11 is based on a new approach to extensions of functions which we call *a lacunary modification* of the Whitney extension method. We present this approach in Subsection 6.2. The main idea of this modification is to use certain *families* of Whitney's cubes rather than to treat each Whitney cube separately. We call these families of Whitney cubes *lacunae*. Each lacuna characterizes a certain "hole" in the complement  $\mathbf{R}^n \setminus E$ .

In Subsection 6.2 we present main definitions and main properties of lacunae. For the proof of these properties we refer the reader to the paper [12], Sections 4-5.

In Subsection 7.1 of **Section 7** we prove several variants of the main result. Let us formulate one of them.

**Theorem 1.6** *Let  $n < p < \infty$  and let  $\mu$  be a non-trivial non-negative Borel measure on  $\mathbf{R}^n$ . A function  $f \in L_{p,loc}(\mathbf{R}^n; \mu)$  belongs to the space  $L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$  if and only if there exists a positive constant  $\lambda$  which satisfies the following conditions for a certain absolute positive constant  $\gamma$ : Let  $\mathcal{Q}$  be an arbitrary finite family of pairwise disjoint cubes in  $\mathbf{R}^n$ . Suppose that to each cube  $Q \in \mathcal{Q}$  we have arbitrarily assigned two cubes  $Q', Q'' \in \mathcal{Q}$  such that  $Q' \cup Q'' \subset \gamma Q$  and*

$$(1.3) \quad (\text{diam } Q')^{p-n} \mu(Q') + (\text{diam } Q'')^{p-n} \mu(Q'') \leq 1.$$

Then the following inequality

$$(1.4) \quad \sum_{Q \in \mathcal{Q}} \left( \frac{\text{diam } Q' \text{ diam } Q''}{\text{diam } Q} \right)^{p-n} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y) \leq \lambda$$

holds. Furthermore,  $\|f\|_{\Sigma} \sim \inf \lambda^{\frac{1}{p}}$  with constants of equivalence depending only on  $n$  and  $p$ .

Note that the hypotheses of this theorem are equivalent to the hypotheses of Theorem 1.3 provided the cubes  $Q', Q''$  from its formulation satisfy inequality (1.3). Thus the sufficiency part of Theorem 1.6 is slightly stronger than the sufficiency part of Theorem 1.3: it asserts that it suffices to verify (1.2) only for cubes satisfying inequality (1.3) rather than for *all* cubes, as required in Theorem 1.3.

In Subsection 7.2, we obtain another variant of Theorem 1.3, which we use in Subsection 7.3 to prove the following explicit formula for calculation of the  $K$ -functional for the couple  $(L_p(\mathbf{R}^n; \mu), L_p^1(\mathbf{R}^n))$ .

**Theorem 1.7** *Let  $n < p < \infty$  and let  $f \in L_{p,loc}(\mathbf{R}^n; \mu)$ . Then, for every  $t > 0$ ,*

$$K(t; f : (L_p(\mathbf{R}^n; \mu), L_p^1(\mathbf{R}^n))) \sim \sup \left\{ \sum_{Q \in \mathcal{Q}} \left( \frac{\text{diam } Q' \text{ diam } Q''}{\text{diam } Q} \right)^{p-n} \frac{\iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)}{(\text{diam } Q')^{p-n} \mu(Q') + (\text{diam } Q'')^{p-n} \mu(Q'')} \right\}^{\frac{1}{p}}$$

where the supremum is taken over all finite families  $\mathcal{Q}$  of pairwise disjoint cubes in  $\mathbf{R}^n$  and all mappings  $\mathcal{Q} \ni Q \mapsto Q' \in \mathcal{Q}$  and  $\mathcal{Q} \ni Q \mapsto Q'' \in \mathcal{Q}$  such that  $Q' \cup Q'' \subset \gamma Q$  and

$$(\text{diam } Q') \left( \frac{\mu(Q')}{|Q'|} \right)^{\frac{1}{p}} + (\text{diam } Q'') \left( \frac{\mu(Q'')}{|Q''|} \right)^{\frac{1}{p}} \leq t.$$

Here  $\gamma$  is an absolute constant. Furthermore, the above equivalence holds with constants depending only on  $n$  and  $p$ .

Using Theorem 1.5 we also prove that this formula for the  $K$ -functional of the couple  $(L_p(\mathbf{R}^n; \mu), L_p^1(\mathbf{R}^n))$  can be quasi-linearized. See Subsection 7.2 for the details.

Finally, in Subsection 7.3 we give a geometrical interpretation of Theorem 3.3 and simple geometrical proofs of some particular cases of it, some of which have been kindly provided by V. Dolnikov.

Our interest in Problems 1.1 and 1.2 has been motivated by their intimate connection with the characterization of the restrictions of Sobolev  $L_p^2(\mathbf{R}^n)$ -functions to arbitrary closed subsets of  $\mathbf{R}^n$ . In particular, Theorem 1.3 is one of the main ingredients of our approach to this problem in [12] where it enables us to give a constructive description of the trace space  $L_p^2(\mathbf{R}^2)|_E$  whenever  $p > 2$  and  $E$  is an arbitrary finite set  $E \subset \mathbf{R}^2$ .

Our second main result here, Theorem 1.5, is also used in [12] in order to prove the existence of a *continuous linear extension operator* from  $L_p^2(\mathbf{R}^2)|_E$  into  $L_p^2(\mathbf{R}^2)$ ,  $p > 2$ , whose operator norm is bounded by a constant depending only on  $p$ . A different proof of this latter result has been given earlier by A. Israel [7]. Quite recently C. Fefferman, A. Israel and

G. K. Luli [5] proved the existence of such an operator for the space  $L_p^m(\mathbf{R}^n)|_E$  whenever  $n < p < \infty$  and  $E \subset \mathbf{R}^n$  is an arbitrary closed set. We refer to [12] for more details.

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## 2. Proof of Theorem 1.3: Necessity.

Throughout the paper  $C, C_1, C_2, \dots$  and  $\gamma, \gamma_1, \gamma_2, \dots$  will be generic positive constants which depend only on  $n$  and  $p$ . Sometimes these constants can depend on certain parameters (say  $\eta, \theta$ , etc.) which we fix in formulations of some auxiliary results. These constants can change even in a single string of estimates. The dependence of a constant on certain parameters is expressed, for example, by the notation  $C = C(n, p)$  or  $\gamma = \gamma(n)$ . We write  $A \sim B$  if there is a constant  $C \geq 1$  such that  $A/C \leq B \leq CA$ .

Throughout the paper the words “a subset of  $\mathbf{R}^n$ ” will mean “a Borel subset of  $\mathbf{R}^n$ ”. For a locally integrable (with respect to the measure  $\mu$ ) function  $f$  and a subset  $S \subset \mathbf{R}^n$  of a positive  $\mu$ -measure by  $f_S$  we denote the  $\mu$ -average of  $f$  over  $S$ :

$$f_S := \frac{1}{\mu(S)} \int_S f d\mu.$$

By  $\|\cdot\|$  we denote the uniform measure in  $\mathbf{R}^n$ . Given a set  $A$  by  $\#A$  we denote the cardinality of  $A$ .

Let  $\mathcal{A}$  be a family of sets in  $\mathbf{R}^n$ . By  $M(\mathcal{A})$  we denote its covering multiplicity, i.e., the minimal positive integer  $M$  such that every point  $x \in \mathbf{R}^n$  is covered by at most  $M$  sets from  $\mathcal{A}$ . Finally, given a function  $g \in L_{1,loc}(\mathbf{R}^n)$  we let  $\mathcal{M}[g]$  denote its Hardy-Littlewood maximal function:

$$(2.1) \quad \mathcal{M}[g](x) := \sup_{K \ni x} \frac{1}{|K|} \int_K g(y) dy, \quad x \in \mathbf{R}^n.$$

As usual, in this formula the supremum is taken over all cubes  $K$  in  $\mathbf{R}^n$  containing  $x$ .

When  $p > n$ , it follows from the Sobolev embedding theorem that every function  $F \in L_p^1(\mathbf{R}^n)$  coincides almost everywhere with a continuous function. This fact enables us to identify each element  $F \in L_p^1(\mathbf{R}^n)$ ,  $p > n$ , with its unique continuous representative.

One of the main tools of the proof of Theorem 1.3 is the following proposition which presents a classical Sobolev imbedding inequality for the case  $p > n$ , see, e.g. [9], p. 61, or [10], p. 55. This inequality is also known in the literature as Sobolev-Poincaré inequality (for  $p > n$ ).

**Proposition 2.1** *Let  $F \in L_p^1(\mathbf{R}^n)$  be a continuous function and let  $n < q \leq p < \infty$ . Then for every cube  $Q \subset \mathbf{R}^n$  and every  $x, y \in Q$  the following inequality*

$$|F(x) - F(y)| \leq C(n, q) \operatorname{diam} Q \left( \frac{1}{|Q|} \int_Q \|\nabla F(z)\|^q dz \right)^{\frac{1}{q}}$$

holds.

We begin the proof of the necessity part with the following auxiliary lemma.

**Lemma 2.2** *Let  $n < q \leq p < \infty$ ,  $\gamma > 1$ , and  $\theta', \theta'' > 0$ . Let  $f = f_1 + f_2$  where  $f_1 \in L_p^1(\mathbf{R}^n)$  and  $f_2 \in L_p(\mathbf{R}^n; \mu)$ . Given a cube  $Q \subset \mathbf{R}^n$  let  $S', S''$  be closed subsets of  $Q$  such that  $S' \cup S'' \subset \gamma Q$ .*

*Then the following inequality*

$$\begin{aligned} \frac{\frac{1}{\mu(S')\mu(S'')} \iint_{S' \times S''} |f(x) - f(y)|^p d\mu(x) d\mu(y)}{(\text{diam } Q)^{p-n} + \theta'/\mu(S') + \theta''/\mu(S'')} &\leq C \left\{ \int_Q \mathcal{M}[(\|\nabla f_1\|)^q]^{\frac{p}{q}}(x) dx \right. \\ &\quad \left. + \frac{1}{\theta'} \int_{S'} |f_2(x)|^p d\mu(x) + \frac{1}{\theta''} \int_{S''} |f_2(x)|^p d\mu(x) \right\} \end{aligned}$$

holds. Here  $C = C(n, p, q, \gamma)$  is a constant depending only  $n, p, q$  and  $\gamma$ .

*Proof.* We have

$$\begin{aligned} I &:= \frac{1}{\mu(S')\mu(S'')} \iint_{S' \times S''} |f(x) - f(y)|^p d\mu(x) d\mu(y) \\ &\leq \frac{1}{\mu(S')\mu(S'')} \iint_{S' \times S''} (|f_1(x) - f_1(y)| + |f_2(x) - f_2(y)|)^p d\mu(x) d\mu(y) \\ &\leq \frac{2^p}{\mu(S')\mu(S'')} \iint_{S' \times S''} |f_1(x) - f_1(y)|^p d\mu(x) d\mu(y) \\ &\quad + \frac{2^p}{\mu(S')\mu(S'')} \iint_{S' \times S''} |f_2(x) - f_2(y)|^p d\mu(x) d\mu(y) = 2^p \{I_1 + I_2\}. \end{aligned}$$

By the Sobolev-Poincaré inequality, see Proposition 2.1, for every  $x, y \in \gamma Q$  we have

$$|f_1(x) - f_1(y)| \leq C(n, q) \text{diam } Q \left( \frac{1}{|\gamma Q|} \int_{\gamma Q} \|\nabla f_1(z)\|^q dz \right)^{\frac{1}{q}}$$

with  $C = C(n, q, \gamma)$ . Hence

$$\begin{aligned} I_1 &:= \frac{1}{\mu(S')\mu(S'')} \iint_{S' \times S''} |f_1(x) - f_1(y)|^p d\mu(x) d\mu(y) \\ &\leq C(\text{diam } Q)^p \left( \frac{1}{|\gamma Q|} \int_{\gamma Q} \|\nabla f_1(z)\|^q dz \right)^{\frac{p}{q}}. \end{aligned}$$



Then, by (2.1), for every  $z \in Q$  we have

$$\left( \frac{1}{|\gamma Q|} \int_{\gamma Q} \|\nabla f_1(z)\|^q dz \right)^{\frac{p}{q}} \leq \mathcal{M}[\|\nabla f_1(z)\|^q]^{\frac{p}{q}}(z).$$

Integrating this inequality on  $Q$  (with respect to  $z$ ) we obtain

$$(2.2) \quad |Q| \left( \frac{1}{|\gamma Q|} \int_{\gamma Q} \|\nabla f_1(z)\|^q dz \right)^{\frac{p}{q}} \leq \int_Q \mathcal{M}[\|\nabla f_1(z)\|^q]^{\frac{p}{q}}(z) dz.$$

Hence

$$(2.3) \quad I_1 \leq C(\text{diam } Q)^{p-n} \int_Q \mathcal{M}[\|\nabla f_1(z)\|^q]^{\frac{p}{q}}(z) dz.$$

Let us estimate the quantity  $I_2$ . We have

$$\begin{aligned} I_2 &:= \frac{1}{\mu(S')\mu(S'')} \iint_{S' \times S''} |f_2(x) - f_2(y)|^p d\mu(x)d\mu(y) \\ &\leq \frac{2^p}{\mu(S')\mu(S'')} \left\{ \iint_{S' \times S''} |f_2(x)|^p d\mu(x)d\mu(y) + \iint_{S' \times S''} |f_2(y)|^p d\mu(x)d\mu(y) \right\} \\ &= 2^p \left\{ \frac{1}{\mu(S')} \int_{S'} |f_2(x)|^p d\mu(x) + \frac{1}{\mu(S'')} \int_{S''} |f_2(x)|^p d\mu(x) \right\}. \end{aligned}$$

Combining this inequality with inequality (2.3) we obtain

$$\begin{aligned} I &= 2^p \{I_1 + I_2\} \leq C \left\{ (\text{diam } Q)^{p-n} \int_Q \mathcal{M}[(\|\nabla f_1\|)^q]^{\frac{p}{q}}(x) dx \right. \\ &\quad \left. + \frac{1}{\mu(S')} \int_{S'} |f_2(x)|^p d\mu(x) + \frac{1}{\mu(S'')} \int_{S''} |f_2(x)|^p d\mu(x) \right\}. \end{aligned}$$

Hence

$$\begin{aligned} I / \{(\text{diam } Q)^{p-n} + \theta' / \mu(S') + \theta'' / \mu(S'')\} &\leq C \left\{ \int_Q \mathcal{M}[(\|\nabla f_1\|)^q]^{\frac{p}{q}}(x) dx \right. \\ &\quad \left. + \frac{1}{\theta'} \int_{S'} |f_2(x)|^p d\mu(x) + \frac{1}{\theta''} \int_{S''} |f_2(x)|^p d\mu(x) \right\} \end{aligned}$$

proving the lemma. ■

We are in a position to prove a slightly more general version of the necessity part of Theorem 1.3.

**Proposition 2.3** *Let  $n < p < \infty, \gamma > 1, N \geq 1$ , and let  $\mu$  be a non-trivial non-negative Borel measure on  $\mathbf{R}^n$ . Let  $\mathcal{Q}$  be a family of cubes in  $\mathbf{R}^n$  with covering multiplicity  $M(\mathcal{Q}) \leq N$ , and let  $\mathcal{S}$  be a finite family of closed subsets of  $\mathbf{R}^n$  of positive  $\mu$ -measure with  $M(\mathcal{S}) \leq N$ . Suppose that to each cube  $Q \in \mathcal{Q}$  we have assigned two subsets  $S'_Q, S''_Q \in \mathcal{S}$  such that*

$$S'_Q \cup S''_Q \subset \gamma Q.$$

*If a function  $f \in L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$ , then*

$$(2.4) \quad \sum_{Q \in \mathcal{Q}} \frac{\frac{1}{\mu(S'_Q)\mu(S''_Q)} \iint_{S'_Q \times S''_Q} |f(x) - f(y)|^p d\mu(x) d\mu(y)}{(\text{diam } Q)^{p-n} \{1 + (\text{diam } S'_Q)^{n-p}/\mu(S'_Q) + (\text{diam } S''_Q)^{n-p}/\mu(S''_Q)\}} \leq \lambda$$

*where  $\lambda = C\|f\|_{\Sigma}^p$ . Here  $C$  is a constant depending only on  $n, p, \gamma$  and  $N$ .*

*Proof.* Let  $q := (p + n)/2$ . Given a cube  $Q \in \mathcal{Q}$  we put  $S' := S'_Q, S'' := S''_Q$ , and

$$\theta' := (\text{diam } Q / \text{diam } S'_Q)^{p-n}, \quad \theta'' := (\text{diam } Q / \text{diam } S''_Q)^{p-n}.$$

Since  $f \in L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$ , there exist functions  $f_1 \in L_p^1(\mathbf{R}^n)$  and  $f_2 \in L_p(\mathbf{R}^n; \mu)$  such that  $f = f_1 + f_2$  and

$$\|f_1\|_{L_p^1(\mathbf{R}^n)} \leq 2\|f\|_{\Sigma}, \quad \|f_2\|_{L_p(\mathbf{R}^n; \mu)} \leq 2\|f\|_{\Sigma}.$$

Then, by Lemma 2.2, the quantity

$$J_Q := \frac{\frac{1}{\mu(S'_Q)\mu(S''_Q)} \iint_{S'_Q \times S''_Q} |f(x) - f(y)|^p d\mu(x) d\mu(y)}{(\text{diam } Q)^{p-n} \{1 + (\text{diam } S'_Q)^{n-p}/\mu(S'_Q) + (\text{diam } S''_Q)^{n-p}/\mu(S''_Q)\}}$$

satisfies the following inequality

$$\begin{aligned} J_Q &\leq C \left\{ \int_Q \mathcal{M}[(\|\nabla f_1\|)^q]^{\frac{p}{q}}(x) dx + \frac{1}{\theta'} \int_{S'} |f_2(x)|^p d\mu(x) + \frac{1}{\theta''} \int_{S''} |f_2(x)|^p d\mu(x) \right\} \\ &= C \left\{ \int_Q \mathcal{M}[(\|\nabla f_1\|)^q]^{\frac{p}{q}}(x) dx \right. \\ &\quad \left. + \left( \frac{\text{diam } S'_Q}{\text{diam } Q} \right)^{p-n} \int_{S'_Q} |f_2(x)|^p d\mu(x) + \left( \frac{\text{diam } S''_Q}{\text{diam } Q} \right)^{p-n} \int_{S''_Q} |f_2(x)|^p d\mu(x) \right\} \\ &= C(J_1 + J_2 + J_3). \end{aligned}$$

Prove that

$$J := \sum_{Q \in \mathcal{Q}} J_Q \leq C\|f\|_{\Sigma}^p.$$

First let us show that the following inequality

$$(2.5) \quad J_1 := \sum_{Q \in \mathcal{Q}} \int_Q \mathcal{M}[(\|\nabla f_1\|)^q]^{\frac{p}{q}}(x) dx \leq C \|f_1\|_{L_p^1(\mathbf{R}^n)}^p$$

holds. In fact, since covering multiplicity of the family  $\mathcal{Q}$  is at most  $N$  and  $p/q > 1$ , by the Hardy-Littlewood maximal theorem,

$$J_1 \leq N \int_{\mathbf{R}^n} \mathcal{M}[\|\nabla f_1\|^q]^{\frac{p}{q}}(x) dx \leq C \int_{\mathbf{R}^n} (\|\nabla f_1\|^q)^{\frac{p}{q}}(x) dx = C \|\nabla f_1\|_{L_p(\mathbf{R}^n)}^p.$$

Prove that

$$(2.6) \quad J_2 := \sum_{Q \in \mathcal{Q}} \left( \frac{\text{diam } S'_Q}{\text{diam } Q} \right)^{p-n} \int_{S'_Q} |f_2(x)|^p d\mu(x) \leq C \|f_2\|_{L_p(\mathbf{R}^n; \mu)}^p.$$

To this end let us fix a set  $S \in \mathcal{S}$  and prove that the quantity

$$I(S) := \sum \{(\text{diam } Q)^{n-p} : Q \in \mathcal{Q}, S'_Q = S\}$$

satisfies the following inequality

$$(2.7) \quad I(S) \leq C(\text{diam } S)^{n-p}.$$

Recall that

$$S = S'_Q \subset \gamma Q \quad \text{for every } Q \in \mathcal{Q},$$

and that  $M(\mathcal{Q}) \leq N$ .

Fix a point  $a \in S$  and put  $K_S := Q(a, \text{diam } S)$ . Define three subfamilies of the family  $\mathcal{Q}$ :

$$\mathcal{Q}_S := \{Q \in \mathcal{Q} : S'_Q = S\},$$

$$\mathcal{Q}_S^{(1)} := \{Q \in \mathcal{Q} : S'_Q = S, Q \cap K_S \neq \emptyset\},$$

and

$$\mathcal{Q}_S^{(2)} := \{Q \in \mathcal{Q} : S'_Q = S, Q \cap K_S = \emptyset\}.$$

Prove that  $\mathcal{Q}_S^{(1)}$  contains at most  $N = N(n, \gamma)$  elements. We will make use of the following simple statement: Let  $Q_1, Q_2$  be cubes in  $\mathbf{R}^n$  such that  $Q_1 \cap Q_2 \neq \emptyset$ . Then the set  $Q_1 \cap (2Q_2)$  contains a cube  $\tilde{Q}$  such that

$$\text{diam } \tilde{Q} \geq \frac{1}{2} \min\{\text{diam } Q_1, \text{diam } Q_2\}.$$

In fact, suppose that  $\text{diam } Q_1 \leq \frac{1}{2} \text{diam } Q_2$ . Since  $Q_1 \cap Q_2 \neq \emptyset$ , we have  $Q_1 \subset 2Q_2$  so that we can put  $\tilde{Q} := Q_1$ .

Assume that  $\text{diam } Q_1 > \frac{1}{2} \text{diam } Q_2$ . Let  $y \in Q_1 \cap Q_2$ . Then there exists a cube  $Q_1^{(y)} \subset Q_1$  such that  $Q_1^{(y)} \ni y$  and

$$\text{diam } Q_1^{(y)} = \frac{1}{2} \text{diam } Q_2.$$

Using the same argument as in the first case we conclude that  $Q_1^{(y)} \subset 2Q_2$ . Thus  $Q_1^{(y)} \subset Q_1 \cap (2Q_2)$  so that we can put  $\tilde{Q} := Q_1^{(y)}$ . Then

$$\text{diam } \tilde{Q} = \frac{1}{2} \text{diam } Q_2 \geq \min\{\text{diam } Q_1, \text{diam } Q_2\}$$

proving the statement.

Let us prove the required inequality

$$(2.8) \quad \# \mathcal{Q}_S^{(1)} \leq C(n, \gamma, N).$$

If a cube  $Q \in \mathcal{Q}_S^{(1)}$ , then  $Q \cap K_S \neq \emptyset$  so that, by the above statement, there exists a cube  $\tilde{Q}$  such that  $\tilde{Q} \subset Q \cap (2K_S)$  and

$$\text{diam } \tilde{Q} \geq \frac{1}{2} \min\{\text{diam } Q, K_S\}.$$

But  $S \subset \gamma Q$  so that  $\text{diam } S \leq \gamma \text{diam } Q$ . Since  $K_S = Q(a, \text{diam } S)$ , we obtain

$$\text{diam } K_S = 2 \text{diam } S \leq 2\gamma \text{diam } Q.$$

Hence

$$\text{diam } \tilde{Q} \geq (1/4\gamma)K_S.$$

Thus  $2K_S \supset \tilde{Q}$  and  $\text{diam } \tilde{Q} \geq (1/4\gamma) \text{diam } K_S$ . Note that the family  $\mathcal{Q}_S^{(1)} \subset \mathcal{Q}$  has covering multiplicity  $M(\mathcal{Q}_S^{(1)}) \leq M(\mathcal{Q}) \leq N$  so that  $M(\{\tilde{Q} : Q \in \mathcal{Q}_S^{(1)}\}) \leq N$  as well. Clearly, the cube  $2K_S$  can contain at most  $C(n, \gamma, N)$  of cubes  $\tilde{Q}$  of diameter at least  $(1/4\gamma) \text{diam } K_S$ . This proves (2.8).

Now we have

$$I_1(S) := \sum \{(\text{diam } Q)^{n-p} : Q \in \mathcal{Q}_S^{(1)}\} \leq (\# \mathcal{Q}_S^{(1)}) \max\{(\text{diam } Q)^{n-p} : Q \in \mathcal{Q}_S^{(1)}\}.$$

Since  $\# \mathcal{Q}_S^{(1)} \leq N(n, \gamma)$  and

$$\text{diam } S \leq \gamma \text{diam } Q, \quad Q \in \mathcal{Q}_S,$$

we conclude that

$$I_1(S) \leq C(n, \gamma)(\text{diam } S)^{n-p}.$$

Let us estimate the quantity

$$I_2(S) := \sum \{(\text{diam } Q)^{n-p} : Q \in \mathcal{Q}_S^{(2)}\}.$$

Recall that  $K_S = Q(a, \text{diam } S)$  where  $a \in S$  and

$$K_S \cap Q \neq \emptyset \quad \text{for every } Q \in \mathcal{Q}_S^{(2)}.$$

Since  $a \in S \subset \gamma Q$  for each  $Q \in \mathcal{Q}_S^{(2)}$  we have

$$\|x - a\| \leq \text{diam}(\gamma Q) = \gamma \text{diam } Q \quad \text{for every } x \in Q.$$

Hence

$$(\text{diam } Q)^{n-p} \leq C(\text{diam } Q)^{-p}|Q| \leq C\|x - a\|^{-p}|Q|, \quad x \in Q.$$

Integrating this inequality over the cube  $Q$  (with respect to  $x$ ), we obtain

$$(\text{diam } Q)^{n-p} \leq C \int_Q \|x - a\|^{-p} dx, \quad Q \in \mathcal{Q}_S^{(2)}.$$

Since  $K_S \cap Q \neq \emptyset$  for every  $Q \in \mathcal{Q}_S^{(2)}$ , we have

$$U_S := \cup\{Q : Q \in \mathcal{Q}_S^{(2)}\} \subset \mathbf{R}^n \setminus K_S.$$

Since  $M(\mathcal{Q}_S^{(2)}) \leq N$ , we obtain

$$\begin{aligned} I_2(S) &:= \sum \{(\text{diam } Q)^{n-p} : Q \in \mathcal{Q}_S^{(2)}\} \leq C \sum_{Q \in \mathcal{Q}_S^{(2)}} \int_Q \|x - a\|^{-p} dx \\ &= C \int_{U_S} \|x - a\|^{-p} dx \leq C \int_{\mathbf{R}^n \setminus K_S} \|x - a\|^{-p} dx \leq C(\text{diam } K_S)^{n-p}. \end{aligned}$$

Since  $\text{diam } K_S \sim \text{diam } S$ , we have

$$I_2(S) \leq C(\text{diam } S)^{n-p}.$$

Finally we obtain

$$(2.9) \quad I(S) = I_1(S) + I_2(S) \leq C(\text{diam } S)^{n-p}$$

proving inequality (2.7).

Using this inequality we have the following estimate of the quantity  $J_2$  defined in (2.6):

$$\begin{aligned} J_2 &= \sum_{Q \in \mathcal{Q}} \left( \frac{\text{diam } S'_Q}{\text{diam } Q} \right)^{p-n} \int_{S'_Q} |f_2(x)|^p d\mu(x) \\ &= \sum_{S \in \mathcal{S}} \sum_{Q \in \mathcal{Q}, S_{Q'}=S} \left( \frac{\text{diam } S}{\text{diam } Q} \right)^{p-n} \int_S |f_2(x)|^p d\mu(x) \\ &= \sum_{S \in \mathcal{S}} (\text{diam } S)^{p-n} \left( \sum_{Q \in \mathcal{Q}, S_{Q'}=S} (\text{diam } Q)^{n-p} \right) \int_S |f_2(x)|^p d\mu(x). \end{aligned}$$

By (2.9),

$$\begin{aligned} J_2 &\leq C \sum_{S \in \mathcal{S}} (\text{diam } S)^{p-n} (\text{diam } S)^{n-p} \int_S |f_2(x)|^p d\mu(x) \\ &\leq C \int_{\mathbf{R}^n} |f_2(x)|^p d\mu(x) = C \|f_2\|_{L_p(\mathbf{R}^n; \mu)}^p. \end{aligned}$$

In the same fashion we prove that

$$J_3 := \sum_{Q \in \mathcal{Q}} \left( \frac{\text{diam } S_Q''}{\text{diam } Q} \right)^{p-n} \int_{S_Q''} |f_2(x)|^p d\mu(x) \leq C \|f_2\|_{L_p(\mathbf{R}^n; \mu)}^p.$$

Finally, summarizing estimates for the quantities  $J_1$ , see (2.5),  $J_2$  and  $J_3$ , we obtain

$$J \leq C(J_1 + J_2 + J_3) \leq C(\|f_1\|_{L_p^1(\mathbf{R}^n)}^p + \|f_2\|_{L_p(\mathbf{R}^n; \mu)}^p) \leq C\|f\|_{\Sigma}^p.$$

The proposition is completely proved. ■

**Theorem 2.4** *Let  $n < p < \infty$ ,  $\gamma > 1$ , and let  $\mu$  be a non-trivial non-negative Borel measure on  $\mathbf{R}^n$ . Let  $\mathcal{Q}$  be a finite family of pairwise disjoint cubes in  $\mathbf{R}^n$ , and let  $\mathcal{S}$  be a finite family of pairwise disjoint closed subsets of  $\mathbf{R}^n$ . Suppose that to each cube  $Q \in \mathcal{Q}$  we have assigned two subsets  $S_Q', S_Q'' \in \mathcal{S}$  such that  $S_Q' \cup S_Q'' \subset \gamma Q$ .*

*If a function  $f \in L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$ , then*

$$\sum_{Q \in \mathcal{Q}} \frac{(\text{diam } Q)^{n-p} \iint_{S_Q' \times S_Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)}{\{(\text{diam } S_Q')^{n-p} + \mu(S_Q')\} \{(\text{diam } S_Q'')^{n-p} + \mu(S_Q'')\}} \leq C \|f\|_{\Sigma}^p.$$

Here  $C$  is a constant depending only on  $n, p$  and  $\gamma$ .

*Proof.* By Proposition 2.3,

$$\sum_{Q \in \mathcal{Q}} (\text{diam } Q)^{n-p} \iint_{S_Q' \times S_Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y) / A(S_Q', S_Q'') \leq C \|f\|_{\Sigma}^p$$

where

$$(2.10) \quad A(S_Q', S_Q'') := \mu(S_Q') \mu(S_Q'') \{1 + (\text{diam } S_Q')^{n-p} / \mu(S_Q') + (\text{diam } S_Q'')^{n-p} / \mu(S_Q'')\}.$$

But

$$A(S_Q', S_Q'') \leq \{(\text{diam } S_Q')^{n-p} + \mu(S_Q')\} \{(\text{diam } S_Q'')^{n-p} + \mu(S_Q'')\},$$

and the theorem follows. ■

Finally, we apply this theorem to a function  $f \in L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$  with  $\mathcal{S} = \mathcal{Q}$ ,  $S_Q' = Q'$ , and  $S_Q'' = Q''$  proving the necessity part of Theorem 1.3.

### 3. A $\mu$ -measure concentration set and $\mu$ -measure concentration cubes.

We turn to the proof of the sufficiency part of Theorem 1.3. Actually, in the next three sections we prove a more general result, Theorem 3.1, which immediately implies the sufficiency in Theorem 1.3.

**Theorem 3.1** *Let  $n < p < \infty$ . A function  $f \in \Sigma = L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$  provided  $f \in L_{p,loc}(\mathbf{R}^n; \mu)$  and there exists a positive constant  $\lambda$  which satisfies the following conditions for a certain absolute positive constant  $\gamma$ : Let  $\mathcal{Q}$  be an arbitrary finite family of pairwise disjoint cubes in  $\mathbf{R}^n$  and let  $\mathcal{Q} \ni Q \mapsto Q' \in \mathcal{Q}$  and  $\mathcal{Q} \ni Q \mapsto Q'' \in \mathcal{Q}$  be arbitrary mappings such that  $Q' \cup Q'' \subset \gamma Q$  and*

$$(3.1) \quad (\text{diam } Q')^{p-n} \mu(Q') + (\text{diam } Q'')^{p-n} \mu(Q'') \leq 1.$$

*Then the following inequality*

$$(3.2) \quad \sum_{Q \in \mathcal{Q}} \left( \frac{\text{diam } Q' \text{ diam } Q''}{\text{diam } Q} \right)^{p-n} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y) \leq \lambda$$

*holds. Furthermore,  $\|f\|_{\Sigma} \leq C(n, p) \lambda^{\frac{1}{p}}$ .*

**Remark 3.2** We first prove a version of Theorem 3.1 where inequality (3.1) is replaced by weaker conditions

$$(3.3) \quad \mu(Q') \leq 2^{32p} (\text{diam } Q')^{n-p} \quad \text{and} \quad \mu(Q'') \leq 2^{32p} (\text{diam } Q'')^{n-p}.$$

It can be readily seen that Theorem 3.1 in its original formulation immediately follows from this weaker variant by transition to the measure  $\tilde{\mu} = 2^{32p} \mu$ .

Thus throughout the proof of the theorem we will assume that the cubes  $Q', Q''$  satisfy inequalities (3.3) rather than (3.1)  $\triangleleft$ .

Let  $f$  be a function on  $\mathbf{R}^n$  satisfying the hypothesis of Theorem 3.1. Let us construct its almost optimal decomposition, i.e., functions  $f_1 \in L_p^1(\mathbf{R}^n)$  and  $f_2 \in L_p(\mathbf{R}^n; \mu)$  with almost minimal norms in the spaces  $L_p^1(\mathbf{R}^n)$  and  $L_p(\mathbf{R}^n; \mu)$  respectively, and such that  $f = f_1 + f_2$ .

We do this in two stages. At the first stage which we present in this section we construct a closed set  $E \subset \mathbf{R}^n$  and a family  $\mathcal{K}_E$  of pairwise disjoint “well separated” cubes of  $\mathbf{R}^n$  with centers in  $E$ , see Proposition 3.5 and definition (3.24). These cubes are determined only by the measure  $\mu$ , and by  $n$  and  $p$ . The set  $E$  and the family  $\mathcal{K}_E$  possess certain  $\mu$ -measure concentration properties which we prove in Corollary 3.9 and Lemma 3.7. In Section 4, using the averages of the function  $f$  on cubes from  $\mathcal{K}_E$  and the Whitney extension method, we define the function  $f_1$ . See (4.5) and (4.6). Finally, we put  $f_2 := f - f_1$ .

We begin the first stage with the following theorem which is an important element of our geometrical construction.

**Theorem 3.3** *Let  $w : \mathbf{R}^n \rightarrow (0, \infty)$  be a positive function on  $\mathbf{R}^n$  such that for every  $x \in \mathbf{R}^n$  the following inequality*

$$(3.4) \quad \liminf_{y \rightarrow x} w(y) > 0$$

*holds. Then there exists a set  $S \subset \mathbf{R}^n$  which satisfies all of the following conditions:*

*(i). For every  $x \in \mathbf{R}^n$  there exists a point  $\tilde{x} \in S$  such that*

$$(3.5) \quad \|x - \tilde{x}\| + w(\tilde{x}) \leq 83 w(x);$$

*(ii). For every  $z_1, z_2 \in S, z_1 \neq z_2$ , we have*

$$w(z_1) + w(z_2) \leq \|z_1 - z_2\|/6.$$

*Proof.* Given an integer  $j$  we define a set

$$(3.6) \quad A_j := \{y \in \mathbf{R}^n : 2^{-j-1} < w(y) \leq 2^{-j}\}.$$

We introduce a metric on  $\mathbf{R}^n$  by letting

$$\rho_w(x, y) := \begin{cases} \|x - y\| + w(x) + w(y), & x \neq y, \\ 0, & x = y. \end{cases}$$

Let  $\varepsilon_j := 14 \cdot 2^{-j}$  and let  $B_j$  be a maximal  $\varepsilon_j$ -net in  $A_j$  with respect to the metric  $\rho_w$ . Thus, if  $A_j \neq \emptyset$  and  $\# B_j > 1$ , the following conditions are satisfied:

(1). For every  $z_1, z_2 \in B_j, z_1 \neq z_2$ , we have

$$\rho_w(z_1, z_2) \geq \varepsilon_j ;$$

(2). For every  $x \in A_j$  there exists a point  $x' \in B_j$  such that

$$\rho_w(x, x') < \varepsilon_j ;$$

Since

$$14w(x) \leq 14 \cdot 2^{-j} = \varepsilon_j < 28w(x) \quad \text{for every } z \in A_j ,$$

we have

$$(3.7) \quad \rho_w(z_1, z_2) \geq 7\{w(z_1) + w(z_2)\}, \quad z_1, z_2 \in B_j ,$$

and

$$(3.8) \quad \rho_w(x, x') < 28w(x)$$

for some  $x' \in B_j$ .

Given  $\varepsilon > 0$  and a set  $B \in \mathbf{R}^n$  we let  $[B]_\varepsilon$  denote the closed  $\varepsilon$ -neighborhood of  $B$  with respect to the metric  $\rho_w$ :

$$[B]_\varepsilon := \{x \in \mathbf{R}^n : \exists y \in B \text{ such that } \rho_w(x, y) \leq \varepsilon\}.$$

Let us define a set  $\tilde{B}_j$  by letting

$$(3.9) \quad \tilde{B}_j := B_j \setminus \left[ \bigcup_{i>j} B_i \right]_{\varepsilon_j} .$$

Finally we put

$$(3.10) \quad S := \bigcup_{j=-\infty}^{\infty} \tilde{B}_j .$$

Prove that  $S$  satisfies all the conditions of the proposition. We do this in three steps.

*The first step.* Prove that  $S \neq \emptyset$ .



Suppose that  $S = \emptyset$  and prove that this contradicts to the condition (3.4). Since  $\bigcup_{j=-\infty}^{\infty} A_j = \mathbf{R}^n$ , there exists  $j_0 \in \mathbb{Z}$  such that  $A_{j_0} \neq \emptyset$ . Hence  $B_{j_0} \neq \emptyset$  as well so that there exists a point  $x_0 \in B_{j_0}$ .

By the assumption

$$S := \bigcup_{j=-\infty}^{\infty} \tilde{B}_j = \emptyset,$$

so that the set

$$\tilde{B}_{j_0} := B_{j_0} \setminus \left[ \bigcup_{i>j_0} B_i \right]_{\varepsilon_{j_0}} = \emptyset.$$

Therefore there exist an integer  $j_1 > j_0$  and a point  $x_1 \in B_{j_1}$  such that

$$\rho_w(x_0, x_1) \leq \varepsilon_{j_0}.$$

But  $\tilde{B}_{j_1} = \emptyset$  as well so that there exist an integer  $j_2 > j_1$  and a point  $x_2 \in B_{j_2}$  such that

$$\rho_w(x_1, x_2) \leq \varepsilon_{j_1}.$$

Continuing this process we get a sequence of points  $\{x_k\}_{k=0}^{\infty}$  such that

$$\rho_w(x_k, x_{k+1}) := \|x_k - x_{k+1}\| + w(x_k) + w(x_{k+1}) \leq \varepsilon_{j_k} := 14 \cdot 2^{-j_k}, \quad k = 0, 1, \dots$$

Hence  $w(x_k) \leq \varepsilon_{j_k}$  so that  $w(x_k) \rightarrow 0$  as  $k \rightarrow \infty$ . Furthermore, since

$$\|x_k - x_{k+1}\| \leq 14 \cdot 2^{-j_k}, \quad k = 0, 1, \dots,$$

$\{x_k\}_{k=0}^{\infty}$  is a Cauchy sequence so that there exist  $\bar{x} \in \mathbf{R}^n$  such that

$$\lim_{k \rightarrow \infty} x_k = \bar{x}.$$

Since  $w \geq 0$ , we obtain

$$\liminf_{x \rightarrow \bar{x}} w(x) = 0,$$

a contradiction.

*The second step.* Prove the property (ii) of the proposition which is equivalent to the inequality

$$(3.11) \quad \rho_w(z_1, z_2) \geq 7\{w(z_1) + w(z_2)\}.$$

Suppose that  $z_1 \neq z_2$  and  $w(z_2) \leq w(z_1)$ . If  $z_1, z_2 \in \tilde{B}_j$  for some integer  $j$ , then (3.11) follows from (3.7). Suppose that  $z_1 \in \tilde{B}_i, z_2 \in \tilde{B}_j$  for some  $i > j$ . Since  $z_2 \in \tilde{B}_j$ , by (3.9),  $z_2 \notin [\tilde{B}_i]_{\varepsilon_j}$  so that

$$\rho_w(z_1, z_2) \geq \varepsilon_j = 14 \cdot 2^{-j}.$$

On the other hand, since  $z_1 \in \tilde{B}_i \subset A_i, z_2 \in \tilde{B}_j \subset A_j$ , by (3.6),

$$w(z_1) \leq 2^{-i}, \quad w(z_2) \leq 2^{-j}.$$

Hence

$$\rho_w(z_1, z_2) \geq 14 \cdot 2^{-j} \geq 7(2^{-j} + 2^{-i}) \geq 7\{w(z_1) + w(z_2)\}$$

proving (3.11).

*The third step.* Prove the property (i) of the proposition. Clearly, if  $x \in S$ , then we can put  $\tilde{x} := x$ .

Let  $x \in \mathbf{R}^n \setminus S$ . Put  $x_0 := x$ . Then there exist an integer  $j_0$  such that  $x \in A_{j_0}$  so that  $x \in A_{j_0} \setminus S$ . By (3.8), there exist a point  $x_0 \in B_{j_0}$  such that

$$(3.12) \quad \rho_w(x_0, x) < 28w(x).$$

If  $x_0 \in \tilde{B}_{j_0}$ , see (3.9), then we put  $\tilde{x} := x_0$  and stop. If  $x_0 \notin \tilde{B}_{j_0}$ , then, by (3.10) and (3.9), there exist an integer  $j_1 > j_0$  and a point  $x_1 \in B_{j_1}$  such that

$$\rho_w(x_0, x_1) \leq \varepsilon_{j_0}.$$

If  $x_1 \in \tilde{B}_{j_1}$ , then we put  $\tilde{x} := x_1$  and stop. If  $x_1 \notin \tilde{B}_{j_1}$ , then there exist an integer  $j_2 > j_1$  and a point  $x_2 \in B_{j_2}$  such that  $\rho_w(x_1, x_2) \leq \varepsilon_{j_1}$ .

We continue this process and after  $k+1$  stages of the procedure we obtain  $k+1$  integers  $j_0 < j_1 < \dots < j_k$  and points  $x_m \in B_{j_m}$ ,  $m = 0, \dots, k$ , such that

$$(3.13) \quad \rho_w(x_m, x_{m+1}) \leq \varepsilon_{j_m}, \quad m = 0, \dots, k-1.$$

If  $x_k \in \tilde{B}_{j_k}$ , then we put  $\tilde{x} := x_k$  and stop. If  $x_k \notin \tilde{B}_{j_k}$ , then, by (3.10) and (3.9), there exist an integer  $j_{k+1} > j_k$  and a point  $x_{k+1} \in B_{j_{k+1}}$  such that  $\rho_w(x_k, x_{k+1}) \leq \varepsilon_{j_k}$ .

Let us prove that this procedure is finite, i.e.,  $x_k \in \tilde{B}_{j_k}$  for some  $k \geq 1$ . In fact, otherwise there exists an infinite sequence of points  $\{x_m\}_{m=0}^\infty$  such that  $x_m \in B_{j_m}$ ,  $m = 0, 1, \dots$ , and

$$\rho_w(x_m, x_{m+1}) := \|x_m - x_{m+1}\| + w(x_m) + w(x_{m+1}) \leq \varepsilon_{j_m} = 14 \cdot 2^{-j_m}.$$

Hence

$$0 \leq w(x_m) \leq 14 \cdot 2^{-j_m}.$$

so that  $w(x_m) \rightarrow 0$  as  $k \rightarrow \infty$ . Furthermore,

$$\|x_m - x_{m+1}\| \leq 14 \cdot 2^{-j_m}$$

so that  $\{x_m\}_{m=0}^\infty$  is a Cauchy sequence. Consequently  $\{x_m\}_{m=0}^\infty$  converges to a point  $\bar{x} \in \mathbf{R}^n$ . Hence

$$\liminf_{x \rightarrow \bar{x}} w(x) = 0$$

which contradicts (3.4).

Thus we have proved that there exists a positive integer  $k$  such that for all  $x_k \in S$  and for all  $m = 0, \dots, k$  inequality (3.13) is satisfied. We have

$$\rho_w(x, x_k) \leq \rho_w(x, x_0) + \sum_{m=0}^{k-1} \rho_w(x_m, x_{m+1})$$

so that, by (3.12) and (3.13),

$$\rho_w(x, x_k) \leq 28w(x) + \sum_{m=0}^{k-1} \varepsilon_{j_m} \leq 28w(x) + 14 \sum_{j \geq j_0} 2^{-j_m} \leq 28w(x) + 28 \cdot 2^{-j_0}.$$

Recall that  $x \in A_{j_0}$  so that  $w(x) \geq 2^{-j_0-1}$ . Hence

$$\rho_w(x, x_k) \leq 28w(x) + 28 \cdot (2w(x)) = 84w(x).$$

Thus the point  $\tilde{x} := x_k \in S$  and the following inequality

$$\rho_w(x, \tilde{x}) := \|x - \tilde{x}\| + w(x) + w(\tilde{x}) \leq 84w(x)$$

holds. This proves inequality (3.5) and the theorem. ■

Note that Theorem 3.3 can be reformulated in a purely geometrical way. We discuss a geometrical background of this theorem and other related geometrical problems in Subsection 7.3.

Fix a point  $x \in \mathbf{R}^n$  and consider two functions of a positive parameter  $r$ : a function

$$s_x(r) := \mu(Q(x, r)), \quad r \in (0, +\infty),$$

and a function

$$v(r) := r^{n-p}, \quad r \in (0, +\infty).$$

Clearly,  $s = s_x(r)$  is a *non-decreasing* function on  $(0, +\infty)$ . Since  $\mu$  is a non-trivial measure on  $\mathbf{R}^n$ ,

$$\lim_{r \rightarrow +\infty} s_x(r) = \mu(\mathbf{R}^n) > 0.$$

On the other hand, since  $p > n$ , the function  $v = v(r)$  is *strictly decreasing* on  $(0, +\infty)$ . Clearly,

$$\lim_{r \rightarrow 0} v(r) = +\infty \quad \text{and} \quad \lim_{r \rightarrow +\infty} v(r) = 0.$$

These properties of the functions  $s_x(r)$  and  $v(r)$  imply *the existence of a unique number*  $R(x) \in (0, \infty)$  such that

$$s_x(r) > v(R(x)) \quad \text{if } r > R(x), \quad \text{and} \quad s_x(r) < v(R(x)), \quad \text{if } r < R(x).$$

Thus, for every  $x \in \mathbf{R}^n$  we have

$$(3.14) \quad \mu(Q(x, r)) > R(x)^{n-p} \quad \text{for every } r > R(x),$$

and

$$(3.15) \quad \mu(Q(x, r)) < R(x)^{n-p} \quad \text{for every } r < R(x).$$

Since  $\mu$  is a Borel measure, the function  $s_x(r) := \mu(Q(x, r))$  is right continuous on  $(0, \infty)$  so that, by (3.14),

$$(3.16) \quad \mu(Q(x, R(x))) \geq R(x)^{n-p}.$$

We also recall that the number  $R(x)$  satisfies the inequality

$$(3.17) \quad 0 < R(x) < +\infty.$$

**Lemma 3.4** *The function  $R = R(x)$  satisfies the Lipschitz condition on  $\mathbf{R}^n$ :*

$$(3.18) \quad |R(x) - R(y)| \leq \|x - y\| \quad \text{for every } x, y \in \mathbf{R}^n.$$

*Proof.* Suppose that  $R(x) > R(y)$ .

Let  $r \in (R(y), R(x))$ . Then, by (3.15),

$$(3.19) \quad \mu(Q(x, r)) < r^{n-p}.$$

Prove that  $Q(y, R(y)) \not\subseteq Q(x, r)$ . In fact, if  $Q(y, R(y)) \subset Q(x, r)$ , then, by (3.16),

$$\mu(Q(x, r)) \geq \mu(Q(y, R(y))) \geq R(y)^{n-p} \geq r^{n-p}$$

which contradicts (3.19).

Thus for every  $r \in (R(y), R(x))$  there exists a point  $a_r \in Q(y, R(y)) \setminus Q(x, r)$  so that  $\|a_r - x\| > r$  and  $\|a_r - y\| \leq R(y)$ . Hence

$$r < \|a_r - x\| \leq \|a_r - y\| + \|y - x\| \leq R(y) + \|x - y\|$$

proving that

$$|r - R(y)| = r - R(y) < \|x - y\|.$$

Since  $r \in (R(y), R(x))$  is arbitrary, we obtain the required inequality (3.18). ■

**Proposition 3.5** *There exists a subset  $E \subset \mathbf{R}^n$  such that:*

(i). *For every  $x, y \in E, x \neq y$ ,*

$$(3.20) \quad 6(R(x) + R(y)) \leq \|x - y\|;$$

(ii). *For every  $x \in \mathbf{R}^n$  there exists a point  $\tilde{x} \in E$  such that*

$$(3.21) \quad R(\tilde{x}) \leq 83 R(x)$$

and

$$(3.22) \quad \|\tilde{x} - x\| \leq 83 R(x).$$

*Proof.* By Lemma 3.4, the function  $R = R(x)$  is Lipschitz continuous on  $\mathbf{R}^n$  so that it is continuous. Hence for every  $x \in \mathbf{R}^n$  we have

$$\liminf_{y \rightarrow x} R(y) = \lim_{y \rightarrow x} R(y) = R(x).$$

Since  $R(x) > 0$  (see (3.17)), condition (3.4) of Theorem 3.3 for the function  $w(x) := R(x), x \in \mathbf{R}^n$ , is satisfied. By this theorem, there exists a set  $E \subset \mathbf{R}^n$  satisfying the required inequalities (3.20), (3.21), and (3.22). ■

Note that, by inequality (3.20), the set  $E$  consists of *isolated points* of  $\mathbf{R}^n$ .

Given  $x \in E$  we let  $K^{(x)}$  denote the cube

$$(3.23) \quad K^{(x)} := Q(x, R(x)).$$

We put

$$(3.24) \quad \mathcal{K}_E := \{K^{(x)} : x \in E\}.$$

Recall that, by the inequality (3.16), for every  $K \in \mathcal{K}_E$  we have

$$(3.25) \quad \mu(K) \geq 2^{p-n}(\text{diam } K)^{n-p}.$$

Let us present several properties of the set  $E$  and the cubes of the family  $\mathcal{K}_E$ .

**Lemma 3.6** (i). *For every two cubes  $K, K' \in \mathcal{K}_E$ ,  $K \neq K'$ , we have*

$$\text{diam } K + \text{diam } K' \leq \text{dist}(K, K')/2;$$

(ii). *Let  $\tau \geq 1$  be a constant and let  $x, x' \in E$ ,  $x \neq x'$ . Let  $Q, Q'$  be cubes in  $\mathbf{R}^n$  such that  $\tau Q \ni x, \tau Q' \ni x'$ , and  $Q \cap Q' \neq \emptyset$ . Then*

$$\text{diam } K^{(x)} + \text{diam } K^{(x')} \leq \tau(\text{diam } Q + \text{diam } Q').$$

*Proof.* (i). Let  $K = K^{(a)}, K' = K^{(a')}$  for some  $a, a' \in E$ . By part (i) of Proposition 3.5,

$$6(R(a) + R(a')) \leq \|a - a'\|.$$

On the other hand

$$\|a - a'\| \leq \text{dist}(K, K') + R(a) + R(a').$$

Hence

$$\begin{aligned} \text{dist}(K, K') &\geq \|a - a'\| - R(a) - R(a') \\ &\geq 6(R(a) + R(a')) - R(a) - R(a') = 5(R(a) + R(a')) \\ &= \frac{5}{2}(\text{diam } K + \text{diam } K') \end{aligned}$$

proving the statement (i).

(ii). By part (i) of Proposition 3.5,

$$3(\text{diam } K^{(x)} + \text{diam } K^{(x')}) \leq \|x - x'\|.$$

Since  $\tau Q \ni x, \tau Q' \ni x'$ , and  $Q \cap Q' \neq \emptyset$ ,

$$\|x - x'\| \leq \tau r_Q + \tau r_{Q'} + r_Q + r_{Q'} = (\tau + 1)(r_Q + r_{Q'}) \leq 2\tau(r_Q + r_{Q'})$$

so that

$$\text{diam } K^{(x)} + \text{diam } K^{(x')} \leq \frac{1}{3}\|x - x'\| \leq \frac{2}{3}\tau(\text{diam } Q + \text{diam } Q').$$

The lemma is proved. ■

**Lemma 3.7** *For every cube  $Q \subset \mathbf{R}^n$  and every  $\theta > 0$  such that*

$$\text{diam } Q \leq \theta \text{dist}(Q, E)$$

*the following inequality*

$$\mu(Q) \leq 42^p(1 + \theta)^p r_Q^{n-p}$$

*holds.*

*Proof.* By Proposition 3.5, for every  $x \in \mathbf{R}^n$  there exists a point  $\tilde{x} \in E$  such that

$$\|\tilde{x} - x\| \leq 83 R(x).$$

Hence,

$$(3.26) \quad \text{dist}(x, E) \leq 83 R(x), \quad x \in \mathbf{R}^n \setminus E.$$

We let  $\lceil \theta \rceil$  denote the (unique) positive integer such that  $\theta \leq \lceil \theta \rceil < \theta + 1$ . Let  $m := 42\lceil \theta \rceil$ . Consider a partition  $\mathcal{K}_Q$  of the cube  $Q$  into  $m^n$  equal cubes  $\{K_1, K_2, \dots, K_{m^n}\}$  of diameter  $\text{diam } Q/m$ . Clearly, for every  $K \in \mathcal{K}_Q$  we have

$$\text{dist}(Q, E) \leq \text{dist}(K, E).$$

Let  $K = Q(c_K, r_K)$ . We have

$$m \text{diam } K = \text{diam } Q \leq \theta \text{dist}(Q, E) \leq \theta \text{dist}(K, E) \leq \theta \text{dist}(c_K, E).$$

By (3.26),

$$\text{dist}(c_K, E) \leq 83 R(c_K)$$

so that

$$m \text{diam } K = 2m r_K \leq \theta \text{dist}(c_K, E) \leq 83 \theta R(c_K).$$

Hence

$$r_K \leq (83 \theta / 2m) R(c_K).$$

Since  $m = 42\lceil \theta \rceil \geq 42 \theta$ , we have  $2m \geq 84 \theta$  so that

$$r_K < R(c_K).$$

Now, by (3.15),

$$\mu(K) = \mu(Q(c_K, r_K)) < R(c_K)^{n-p}$$

so that

$$\mu(K) < r_K^{n-p} = (r_Q/m)^{n-p}.$$

Hence,

$$\mu(Q) = \sum_{i=1}^{m^n} \mu(K_i) \leq m^n m^{p-n} r_Q^{n-p} = m^p r_Q^{n-p} \leq 42^p (1 + \theta)^p r_Q^{n-p}.$$

The lemma is proved. ■

The next lemma states that an inequality which is converse to the inequality (3.16) is also true.

**Lemma 3.8** *For every  $x \in E$  the following inequality*

$$\mu(Q(x, 5R(x))) \leq 2^{14p} R(x)^{n-p}$$

*holds.*

*Proof.* By subdividing each edge of the cube  $K^{(x)} = Q(x, 5R(x))$  into 20 equal parts we can partition this cube into a family  $A$  consisting of  $20^n$  congruent cubes of diameter  $\frac{1}{4} \text{diam } K^{(x)}$ . Clearly, those cubes of the family  $A$  which contain the point  $x$ , the center of the cube  $K^{(x)}$ , are a partition of the cube  $\frac{1}{2}K^{(x)}$  into a family of  $2^n$  congruent cubes.

Thus the set  $5K^{(x)} \setminus (\frac{1}{2}K^{(x)})$  is partitioned into a family  $B \subset A$  consisting of  $20^n - 2^n$  congruent cubes of diameter  $\frac{1}{4} \text{diam } K^{(x)}$ . Clearly, for each cube  $K \in B$

$$(3.27) \quad \text{dist}(K, \{x\}) \geq \text{diam } K \left( = \frac{1}{4} \text{diam } K^{(x)} \right).$$

On the other hand, by part (i) of Lemma 3.5, the family

$$6\mathcal{K}_E := \{6K^{(y)} = Q(x, 6R(y)) : y \in E\}$$

consists of *non-overlapping* cubes. Hence

$$\text{dist}(5K^{(x)}, E \setminus (5K^{(x)})) \geq R(x) = \frac{1}{2} \text{diam } K^{(x)} = 2 \text{diam } K.$$

Since  $K \subset 5K^{(x)}$ ,

$$\text{dist}(K, E \setminus (5K^{(x)})) \geq \text{dist}(5K^{(x)}, E \setminus (5K^{(x)})) \geq 2 \text{diam } K.$$

Combining this inequality with (3.27) we obtain

$$\text{dist}(K, E) \geq \text{diam } K.$$

This property of the cube  $K$  enables us to apply to  $K$  the result of Lemma 3.7 with  $\theta = 1$ . By this lemma,

$$\mu(K) \leq 84^p r_K^{n-p}.$$

Since  $r_K = \frac{1}{4} R(x)$ , we have

$$\mu(K) \leq 84^p \left( \frac{1}{4} R(x) \right)^{n-p} = 4^{p-n} 84^p R(x)^{n-p}.$$

By inequality (3.15),

$$\mu\left(\frac{1}{2}K^{(x)}\right) = \mu\left(Q\left(x, \frac{1}{2}R(x)\right)\right) < R(x)^{n-p}.$$

Finally we have

$$\mu(5K^{(x)}) \leq \mu\left(\frac{1}{2}K^{(x)}\right) + \sum_{K \in B} \mu(K)$$

so that

$$\mu(5K^{(x)}) \leq R(x)^{n-p} + (20^n - 2^n) 4^{p-n} 84^p R(x)^{n-p} \leq 2^{14p} R(x)^{n-p}$$

proving the lemma. ■

This lemma and inequality (3.25) imply the following

**Corollary 3.9** *For every cube  $K \in \mathcal{K}_E$  we have*

$$(3.28) \quad 2^{p-n} (\text{diam } K)^{n-p} \leq \mu(K) \leq 2^{15p} (\text{diam } K)^{n-p}$$

and

$$\mu(5K) \leq 2^{14p} \mu(K).$$

#### 4. Sufficiency: the Sobolev norm of the function $f_1$ .

In this section, given a function  $f$  satisfying the sufficiency condition of Theorem 3.1, we define functions  $f_1 \in L_p^1(\mathbf{R}^n)$  and  $f_2 \in L_p(\mathbf{R}^n; \mu)$  such that  $f_1 + f_2 = f$ . We prove that

$$\|f_1\|_{L_p^1(\mathbf{R}^n)} \leq C\lambda^{\frac{1}{p}}$$

where  $\lambda$  is the constant from inequality (3.2) and  $C = C(n, p)$ . In the next section we show that  $\|f_2\|_{L_p(\mathbf{R}^n; \mu)} \leq C\lambda^{\frac{1}{p}}$ .

Let  $E$  be the set constructed in the previous section. Since  $E$  is a closed set, the set  $\mathbf{R}^n \setminus E$  is open so that it admits a Whitney decomposition  $W_E$  into a family of non-overlapping cubes. In the next theorem we recall the main properties of this decomposition. See, e.g. [13], or [6].

**Theorem 4.1**  $W_E = \{Q_k\}$  is a countable family of non-overlapping cubes such that

- (i).  $\mathbf{R}^n \setminus E = \cup\{Q : Q \in W_E\}$ ;
- (ii). For every cube  $Q \in W_E$  we have

$$(4.1) \quad \text{diam } Q \leq \text{dist}(Q, E) \leq 4 \text{diam } Q.$$

Let us note an important property of the Whitney cubes.

**Lemma 4.2** For every cube  $Q \in W_E$  the following inequality

$$\mu(Q) \leq 84^p r_Q^{n-p}$$

holds.

*Proof.* Since  $Q \in W_E$ , by Theorem 4.1,  $\text{diam } Q \leq \text{dist}(Q, E)$ . It remains to apply Lemma 3.7 with  $\theta = 1$ , and the lemma follows.  $\blacksquare$

Combining this result with the second inequality in (3.28) we obtain the following

**Corollary 4.3** Every cube  $Q \in W_E \cup \mathcal{K}_E$  satisfies the inequality

$$(4.2) \quad \mu(Q) \leq 2^{15p} (\text{diam } Q)^{n-p}.$$

Thus for every  $Q', Q'' \in W_E \cup \mathcal{K}_E$  the conditions (3.3) of Theorem 3.1 hold.

We are also needed certain additional properties of Whitney cubes which we present in the next lemma. These properties easily follow from constructions of Whitney decomposition presented in [13] and [6].

Given a cube  $Q \subset \mathbf{R}^n$  let  $Q^* := \frac{9}{8}Q$ .

**Lemma 4.4** (1). If  $Q, K \in W_E$  and  $Q^* \cap K^* \neq \emptyset$ , then

$$\frac{1}{4} \text{diam } Q \leq \text{diam } K \leq 4 \text{diam } Q.$$

(2). For every cube  $K \in W_E$  there are at most  $N = N(n)$  cubes from the family  $W_E^* := \{Q^* : Q \in W_E\}$  which intersect  $K^*$ .

(3). If  $Q, K \in W_E$ , then  $Q^* \cap K^* \neq \emptyset$  if and only if  $Q \cap K \neq \emptyset$ .



Note that inequality (4.1) implies the following property of Whitney cubes:

$$(4.3) \quad (9Q) \cap E \neq \emptyset \quad \text{for every } Q \in W_E.$$

Let us fix a constant  $\tau \geq 9$ . Then by the above property

$$(\tau Q) \cap E \neq \emptyset.$$

To every cube  $Q \in W_E$  we assign a point  $a_Q \in E$  such that

$$(4.4) \quad a_Q \in \tau Q.$$

For instance, one can choose  $a_Q$  to be a point nearest to  $Q$  on the set  $E$ . Then, by the property (4.3),  $a_Q \in \tau Q$  with  $\tau = 9$ .

Let  $\Phi_E := \{\varphi_Q : Q \in W_E\}$  be a smooth partition of unity subordinated to the Whitney decomposition  $W_E$ . Recall the main properties of this partition.

**Lemma 4.5** *The family of functions  $\Phi_E$  has the following properties:*

- (a).  $\varphi_Q \in C^\infty(\mathbf{R}^n)$  and  $0 \leq \varphi_Q \leq 1$  for every  $Q \in W_E$ ;
- (b).  $\text{supp } \varphi_Q \subset Q^*(:= \frac{9}{8}Q)$ ,  $Q \in W_E$ ;
- (c).  $\sum \{\varphi_Q(x) : Q \in W_E\} = 1$  for every  $x \in \mathbf{R}^n \setminus S$ ;
- (d).  $\|\nabla \varphi_Q(x)\| \leq C(n)/\text{diam } Q$  for every  $Q \in W_E$  and every  $x \in \mathbf{R}^n$ .

We turn to definition of the functions  $f_1 \in L_p^1(\mathbf{R}^n)$  and  $f_2 \in L_p(\mathbf{R}^n; \mu)$  which provides an almost optimal decomposition of a function  $f$  satisfying the sufficiency condition of Theorem 1.3.

Let  $\tilde{f}_1 : E \rightarrow \mathbf{R}$  be a function defined by the following formula:

$$(4.5) \quad \tilde{f}_1(x) := f_Q = \frac{1}{\mu(Q)} \int_Q f d\mu \quad \text{for every } x \in E.$$

Here  $Q = K^{(x)} = Q(x, R(x))$ , see (3.23).

Using the Whitney extension formula we extend  $\tilde{f}_1$  from  $E$  to all of  $\mathbf{R}^n$ . We denote this extension by  $f_1$ . Thus:

$$(4.6) \quad f_1(x) := \begin{cases} \tilde{f}_1(x), & x \in E, \\ \sum_{Q \in W_E} \varphi_Q(x) \tilde{f}_1(a_Q), & x \in \mathbf{R}^n \setminus E. \end{cases}$$

Finally we put

$$f_2 := f - f_1.$$

Our goal is to prove that under Theorem's 3.1 conditions the following inequality

$$(4.7) \quad \|f_1\|_{L_p^1(\mathbf{R}^n)} + \|f_2\|_{L_p(\mathbf{R}^n; \mu)} \leq C\lambda^{\frac{1}{p}}$$

holds. Here  $C$  is a constant depending only on  $n, p$ , and  $\tau$ .

Let us estimate the norm  $\|f_1\|_{L_p^1(\mathbf{R}^n)}$ . Let  $K$  be a cube in  $\mathbf{R}^n$  and let

$$V_K := \{Q \in W_E : Q \cap K \neq \emptyset\}.$$

**Lemma 4.6** . For every cube  $K \in W_E$  the following inequality

$$\int_K \|\nabla f_1(x)\|^p dx \leq C(n) \sum_{Q \in V_K} \frac{|\tilde{f}_1(a_K) - \tilde{f}_1(a_Q)|^p}{(\text{diam } K)^{p-n}}$$

holds.

*Proof.* Let  $x \in K$ . Since  $x \in \mathbf{R}^n \setminus E$ , by the extension formula (4.6) and by properties (b) and (c) of Lemma 4.5, we have

$$\begin{aligned} \|\nabla f_1(x)\| &= \|\nabla(f_1(x) - \tilde{f}_1(a_K))\| = \left\| \nabla \left( \sum_{Q \in W_E} \varphi_Q(x)(\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)) \right) \right\| \\ &= \left\| \sum_{Q \in W_E} (\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)) \nabla \varphi_Q(x) \right\| \\ &= \left\| \sum \left\{ (\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)) \nabla \varphi_Q(x) : Q^* \cap K \neq \emptyset, Q \in W_E \right\} \right\| \\ &\leq \sum \left\{ |\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)| \|\nabla \varphi_Q(x)\| : Q^* \cap K \neq \emptyset, Q \in W_E \right\}. \end{aligned}$$

Hence, by property (d) of Lemma 4.5,

$$\|\nabla f_1(x)\| \leq C \sum \left\{ \frac{|\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)|}{\text{diam } Q} : Q^* \cap K \neq \emptyset, Q \in W_E \right\}.$$

By Lemma 4.4,  $Q^* \cap K \neq \emptyset$  iff  $Q \cap K \neq \emptyset$ . Also, by this lemma,  $\text{diam } Q \sim \text{diam } K$ . Hence

$$\|\nabla f_1(x)\| \leq C \sum \left\{ \frac{|\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)|}{\text{diam } K} : Q \cap K \neq \emptyset, Q \in W_E \right\}.$$

Integrating this inequality over the cube  $K$ , we obtain

$$\int_K \|\nabla f_1(x)\|^p dx \leq C \sum \left\{ \frac{|\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}} : Q \cap K \neq \emptyset, Q \in W_E \right\}$$

proving the lemma. ■

Recall that the set  $E$  consists of *isolated points* of  $\mathbf{R}^n$  so that the function  $f_1 \in C^\infty(\mathbf{R}^n)$ . This observation and Lemma 4.6 enable us to estimate its Sobolev seminorm as follows:

$$(4.8) \quad \|\nabla f_1\|_{L_p(\mathbf{R}^n)}^p \leq C(n) \sum_{K \in W_E} \sum_{Q \in V_K} \frac{|\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}}.$$

Let us slightly simplify this inequality. By  $\tilde{K}$  we denote a cube which maximize the quantity  $|\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)|$  on the family  $V_K$ ; thus

$$(4.9) \quad \max_{Q \in V_K} |\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)| = |\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|.$$

(Of course,  $\tilde{K}$  depends on  $K$  and  $f$ .) By part (2) of Lemma 4.4,  $\#V_K \leq N(n)$  so that, by (4.8),

$$(4.10) \quad \|\nabla f_1\|_{L_p(\mathbf{R}^n)}^p \leq C(n) \sum_{K \in W_E} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}}.$$

Let us show that we can omit in the right hand side of this inequality those cubes  $K \in W_E$  which lie inside of cubes from the family  $\mathcal{K}_E$ .

Let

$$(4.11) \quad \eta := \frac{1}{21} \tau.$$

**Lemma 4.7** *Let  $x \in E$  and let  $K \in W_E$  be a cube such that*

$$(4.12) \quad K \cap (\eta K^{(x)}) \neq \emptyset.$$

*Then*

$$\tau Q \subset K^{(x)} \quad \text{for every } Q \in V_K.$$

*Furthermore,  $a_Q = x$  for every  $Q \in V_K$ .*

*Proof.* Let  $K = Q(x_K, r_K)$  and let  $Q = Q(x_Q, r_Q)$ . Since  $K \in W_E$ , we have

$$\text{diam } K \leq 4 \text{dist}(K, E).$$

Hence  $\text{diam } K \leq 4 \text{dist}(K, \{x\})$  so that, by (4.12),

$$(4.13) \quad \text{diam } K \leq 4\left(\frac{1}{2} \text{diam}(\eta K^{(x)})\right) = 2\eta \text{diam } K^{(x)}.$$

Since  $Q \in V_K$ , we have  $Q \cap K \neq \emptyset$  and  $Q \in W_E$ , so that, by Lemma 4.4,

$$\text{diam } Q \leq 4 \text{diam } K.$$

Hence

$$(4.14) \quad \text{diam } Q \leq 8\eta \text{diam } K^{(x)}.$$

Furthermore, by (4.12) and (4.13),

$$\begin{aligned} \|x - x_K\| &\leq \frac{1}{2} \text{diam}(\eta K^{(x)}) + \frac{1}{2} \text{diam } K \\ &\leq (\eta/2) \text{diam } K^{(x)} + \frac{1}{2}(2\eta \text{diam } K^{(x)}) = \frac{3}{2}\eta \text{diam } K^{(x)} \end{aligned}$$

so that

$$K \subset (3\eta + 2\eta)K^{(x)} = 5\eta K^{(x)}.$$

Hence

$$\tau K \subset 5\eta \tau K^{(x)} \subset K^{(x)}.$$

(Recall that  $\eta = 1/(21\tau)$ , see (4.11).) Since the cubes of the family  $\mathcal{K}_E = \{K^{(x)} : x \in E\}$  are pairwise disjoint,

$$(\tau K) \cap E \subset K^{(x)} \cap E = \{x\}.$$

Hence  $a_K = x$ .

In the same fashion we show that  $a_Q = x$ . In fact, by (4.14),

$$\begin{aligned} \|x - x_Q\| &\leq \|x - x_K\| + \|x_K - x_Q\| \leq \frac{3}{2}\eta \operatorname{diam} K^{(x)} + \frac{1}{2} \operatorname{diam} K + \frac{1}{2} \operatorname{diam} Q \\ &\leq \frac{3}{2}\eta \operatorname{diam} K^{(x)} + \frac{1}{2}(2\eta \operatorname{diam} K^{(x)}) + \frac{1}{2}(8\eta \operatorname{diam} K^{(x)}) = \frac{13}{2}\eta \operatorname{diam} K^{(x)}. \end{aligned}$$

This inequality and (4.14) imply the following:

$$Q \subset (13\eta + 8\eta)K^{(x)} = 21\eta K^{(x)}.$$

Hence

$$\tau Q \subset 21\tau\eta K^{(x)} \subset K^{(x)}$$

so that

$$(\tau Q) \cap E \subset K^{(x)} \cap E = \{x\}.$$

Thus  $a_Q = a_K = x$ , and the proof is finished. ■

The lemma motivates us to introduce two subsets of  $\mathbf{R}^n$  defined by the following formulas:

$$(4.15) \quad T_E := \bigcup_{x \in E} K^{(x)} = \bigcup \{K : K \in \mathcal{K}_E\}$$

and

$$(4.16) \quad T_{E,\tau} := \bigcup_{x \in E} \eta K^{(x)} = \bigcup \{\eta K : K \in \mathcal{K}_E\}.$$

Let us also introduce a collection of cubes

$$(4.17) \quad \mathcal{A} := \{K \in W_E : K \cap T_{E,\tau} = \emptyset\}.$$

By the lemma

$$\frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\operatorname{diam} K)^{p-n}} = 0$$

provided  $K \cap T_{E,\tau} \neq \emptyset$  or, equivalently,  $K \in W_E \setminus \mathcal{A}$ . Combining this with inequality (4.10) we obtain the following

**Corollary 4.8** *We have*

$$\|\nabla f_1\|_{L_p(\mathbf{R}^n)}^p \leq C(n) \sum_{K \in \mathcal{A}} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\operatorname{diam} K)^{p-n}}.$$

We introduce two subfamilies of the family  $\mathcal{A}$ :

$$(4.18) \quad \mathcal{A}_1 := \{K \in W_E : K \cap T_E = \emptyset\}.$$

and

$$\mathcal{A}_2 := \{K \in W_E : K \cap T_E \neq \emptyset, K \cap T_{E,\tau} = \emptyset\}.$$

Clearly,  $\mathcal{A}_1, \mathcal{A}_2$  is a partition of  $\mathcal{A}$ , i.e.,  $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$ ,  $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$ . Hence, by Corollary 4.8,

$$\begin{aligned} \|\nabla f_1\|_{L_p(\mathbf{R}^n)}^p &\leq C(n) \sum_{K \in \mathcal{A}} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}} \\ &= C(n) \sum_{K \in \mathcal{A}_1} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}} + C(n) \sum_{K \in \mathcal{A}_2} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}}. \end{aligned}$$

Recall that the function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  satisfies the sufficiency condition of Theorem 3.1. Thus there exists a constant  $\lambda > 0$  such that for every finite family  $\mathcal{Q}$  of pairwise disjoint cubes and arbitrary mappings  $\mathcal{Q} \ni Q \mapsto Q' \in \mathcal{Q}$  and  $\mathcal{Q} \ni Q \mapsto Q'' \in \mathcal{Q}$  such that  $Q' \cup Q'' \subset \gamma Q$ , inequality (3.2) holds.

**Lemma 4.9** *Let  $Q \in \mathcal{A}$ . Then for every cube  $Q' \in W_E$  such that  $Q' \cap Q \neq \emptyset$  we have*

$$K^{(a_{Q'})} \subset (22\tau^2)Q.$$

*Proof.* Recall that  $\eta = 1/(21\tau)$ , see (4.11), and

$$T_{E,\tau} = \{Q \in W_E : Q \cap (\eta K^{(x)}) = \emptyset \text{ for every } x \in E\},$$

see (4.16). Since  $Q \in \mathcal{A}$ , we have  $Q \cap T_{E,\tau} = \emptyset$  so that

$$\text{dist}(a_Q, Q) > \frac{1}{2}\eta \text{diam } K^{(a_Q)}.$$

Let  $Q = Q(x_Q, r_Q)$ ,  $Q' = Q(x_{Q'}, r_{Q'})$ . Since  $a_Q \in \tau Q$ , we have  $\|a_Q - x_Q\| \leq \tau r_Q$  so that

$$\text{dist}(K^{(a_Q)}, Q) \leq \|a_Q - x_Q\| \leq \tau r_Q$$

proving that

$$\text{diam } K^{(a_Q)} \leq 2\tau r_Q / \eta = (21\tau^2) \text{diam } Q.$$

Hence

$$K^{(a_Q)} \subset (\tau + 21\tau^2)Q \subset 22\tau^2 Q.$$

Now if  $a_{Q'} = a_Q$ , then  $K^{(a_{Q'})} = K^{(a_Q)}$  so that in this case  $K^{(a_{Q'})} = K^{(a_Q)} \subset 22\tau^2 Q$ .

Suppose that  $a_{Q'} \neq a_Q$ . Then, by part (ii) of Lemma 3.6,

$$(4.19) \quad \text{diam } K^{(a_{Q'})} \leq \tau(\text{diam } Q + \text{diam } Q') \leq \tau(\text{diam } Q + 4 \text{diam } Q) = 10\tau r_Q.$$

Since  $a_{Q'} \in \tau Q'$ , we have  $\|a_{Q'} - x_{Q'}\| \leq \tau r_{Q'}$ . Since  $Q \cap Q' \neq \emptyset$  and  $r_{Q'} \leq 4r_Q$ ,

$$\|x_{Q'} - x_Q\| \leq r_Q + r_{Q'} \leq 5r_Q$$

so that

$$\|a_{Q'} - x_Q\| \leq 5r_Q + \tau r_{Q'} \leq (4\tau + 5)r_Q.$$

Combining this inequality with (4.19) we obtain

$$K^{(a_{Q'})} \subset (4\tau + 5 + 10\tau)Q \subset 22\tau^2 Q.$$

The lemma is proved. ■

We are needed the following combinatorial

**Proposition 4.10** *Let  $N \in \mathbf{N}$  and let  $\mathcal{B} = \{Q\}$  be a collection of cubes in  $\mathbf{R}^n$ . Suppose that for every cube  $Q \in \mathcal{B}$  there exist at most  $N$  cubes from  $\mathcal{B}$  which have common points with  $Q$ . Then the family  $\mathcal{B}$  can be partitioned into at most  $N + 1$  families of pairwise disjoint cubes.*

*Proof.* The proposition immediately follows from the next well-known result in the graph theory (see, e.g., [8]): *Every graph can be colored with one more color than the maximum vertex degree.* ■

**Proposition 4.11** *Suppose that the hypothesis of Theorem 3.1 holds with  $\gamma = 22\tau^2$ . Then*

$$(4.20) \quad \sum_{K \in \mathcal{A}_1} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}} \leq C(n)\lambda.$$

*Proof.* Let  $\tilde{\mathcal{A}} \subset \mathcal{A}_1$  be a finite family of pairwise disjoint cubes. Prove that

$$(4.21) \quad \sum_{K \in \tilde{\mathcal{A}}} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}} \leq \lambda$$

Let

$$(4.22) \quad \mathcal{Q} := \tilde{\mathcal{A}} \cup \mathcal{K}_E.$$

Since the cubes of the family  $\mathcal{A}_1$  and the cubes of the family  $\mathcal{K}_E$  have no common points, see (4.15) and (4.18), the cubes of the family  $\mathcal{Q}$  are pairwise disjoint. Furthermore, these cubes satisfy inequality (4.2).

Let  $Q \in \mathcal{Q}$ . We define two cubes  $Q', Q'' \in \mathcal{Q}$ ,  $Q' \cup Q'' \subset \gamma Q$ , as follows. If  $Q = K \in \tilde{\mathcal{A}}$ , we put

$$(4.23) \quad Q' := K^{(a_K)}, \quad Q'' := K^{(a_{\tilde{K}})}.$$

Then, by definition (4.5),

$$\tilde{f}_1(a_K) := f_{Q'}, \quad \tilde{f}_1(a_{\tilde{K}}) := f_{Q''}.$$

Furthermore, by Lemma 4.9,  $Q' \cup Q'' \subset \gamma Q$  with  $\gamma = 22\tau^2$ .

If  $Q \in \mathcal{K}_E$ , i.e.,  $Q = K^{(x)}$  for some  $x \in E$ , we put  $Q' = Q'' := Q$ . Clearly in this case  $Q' \cup Q'' \subset \gamma Q$  with  $\gamma = 1$ , and  $f_{Q'} = f_{Q''}$ .

By these equalities,

$$I := \sum_{K \in \tilde{\mathcal{A}}} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}} = \sum_{Q \in \tilde{\mathcal{A}}} \frac{|f_{Q'} - f_{Q''}|^p}{(\text{diam } Q)^{p-n}} + \sum_{Q \in \mathcal{K}_E} \frac{|f_{Q'} - f_{Q''}|^p}{(\text{diam } Q)^{p-n}},$$

so that

$$(4.24) \quad I = \sum_{Q \in \mathcal{Q}} \frac{|f_{Q'} - f_{Q''}|^p}{(\text{diam } Q)^{p-n}}.$$

Since  $Q', Q'' \in \mathcal{K}_E$  for every  $Q \in \mathcal{Q}$ , by (3.25),

$$\mu(Q') \geq 2^{p-n}(\text{diam } Q')^{n-p}, \quad \mu(Q'') \geq 2^{p-n}(\text{diam } Q'')^{n-p}$$

proving that

$$(4.25) \quad \frac{1}{\mu(Q')\mu(Q'')} \leq 2^{2(n-p)}(\text{diam } Q' \text{ diam } Q'')^{p-n} \leq (\text{diam } Q' \text{ diam } Q'')^{p-n}.$$

Also

$$(4.26) \quad |f_{Q'} - f_{Q''}|^p \leq \frac{1}{\mu(Q')\mu(Q'')} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y).$$

Combining this inequality with (4.24) and (4.25), we obtain

$$I \leq \sum_{Q \in \mathcal{Q}} \left( \frac{\text{diam } Q' \text{ diam } Q''}{\text{diam } Q} \right)^{p-n} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y).$$

By the assumption (see inequality (3.2)) and in view of Corollary 4.3,  $I \leq \lambda$  proving inequality (4.21).

Since all the terms of the sum in the left hand side of (4.21) are non-negative, this inequality holds for an *arbitrary* (not necessarily finite) subfamily  $\tilde{\mathcal{A}}$  of  $\mathcal{A}_1$  consisting of pairwise disjoint cubes.

To prove inequality (4.21) for the family  $\mathcal{A}_1$  itself (and consequently to prove the proposition) it remains to make use of Proposition 4.10. In fact, every Whitney cube touches at most  $N(n)$  Whitney cubes, see part (2) of Lemma 4.4. Since  $\mathcal{A}_1 \subset W_E$ , the same is true for cubes of the family  $\mathcal{A}_1$ . Hence, by Proposition 4.10,  $\mathcal{A}_1$  can be partitioned into at most  $N(n) + 1$  families of pairwise disjoint cubes. Applying to every such a family inequality (4.21) we obtain the required estimate (4.20).

The proposition is proved. ■

Let us prove an analog of Proposition 4.11 for the family  $\mathcal{A}_2$ . Recall that

$$\mathcal{A}_2 = \{Q \in W_E : Q \cap K \neq \emptyset \text{ for some } K \in \mathcal{K}_E, \text{ and } Q \cap (\eta H) = \emptyset \text{ for every } H \in \mathcal{K}_E\}.$$

Let us fix a cube  $K \in \mathcal{K}_E$  and consider a family of cubes

$$J_K := \{Q \in \mathcal{A}_2 : Q \cap K \neq \emptyset\}.$$

Thus

$$J_K := \{Q \in W_E : Q \cap K \neq \emptyset, Q \cap (\eta H) = \emptyset \text{ for every } H \in \mathcal{K}_E\}.$$

**Lemma 4.12** (i). If  $K, K' \in \mathcal{K}_E$  and  $K \neq K'$ , then  $(5K) \cap (5K') = \emptyset$ ;

(ii). Let  $Q \in W_E, K \in \mathcal{K}_E$ , and let  $Q \cap K \neq \emptyset$ . Then  $\text{diam } Q \leq 2 \text{diam } K$  and  $Q \subset 5K$ ;

(iii). Let  $K, K' \in \mathcal{K}_E$  and let  $Q \in W_E$ . Suppose that  $K \cap Q \neq \emptyset$  and  $K' \cap Q \neq \emptyset$ . Then  $K = K'$ .

*Proof.* (i). This property immediately follows from part (i) of Proposition 3.5, see (3.20).

(ii). Let  $K = K^{(x)}$  for some  $x \in E$ . Since  $Q \cap K \neq \emptyset$ , we have  $\text{dist}(Q, E) \leq r_K$ . Since  $Q \in W_E$ , we have  $\text{diam } Q \leq 4 \text{dist}(Q, E)$  so that

$$\text{diam } Q \leq 4 \text{dist}(Q, E) \leq 4r_K = 2 \text{diam } K.$$

Now let  $z \in Q$ . Since  $Q \cap K \neq \emptyset$ ,

$$\|z - x\| \leq r_K + \text{diam } Q \leq r_K + 4r_K = 5r_K$$

proving the required inclusion  $Q \subset 5K$ .

(iii). If  $K \neq K'$ , then, by part (ii) of the lemma  $Q \subset 5K$  and  $Q \subset 5K'$  so that  $(5K) \cap (5K') \neq \emptyset$  which contradicts the property (i).

The lemma is proved. ■

**Corollary 4.13** (i). For every  $K \in \mathcal{K}_E$  we have

$$\bigcup_{Q \in J_K} Q \subset 5K;$$

(ii).  $J_K \cap J_{K'} = \emptyset$  provided  $K, K' \in \mathcal{K}_E$  and  $K \neq K'$ .

**Lemma 4.14** For every cube  $K \in \mathcal{A}_2$

$$(4.27) \quad \#J_K \leq (45/\eta)^n.$$

In addition for every cube  $Q \in J_K$

$$(4.28) \quad \frac{1}{2} \text{diam } Q \leq \text{diam } K \leq 9\eta \text{diam } Q.$$

*Proof.* Note that, by definition of the family  $J_K$ , for every  $Q \in J_K$  we have

$$(4.29) \quad Q \cap K \neq \emptyset \quad \text{and} \quad Q \cap (\eta K) = \emptyset.$$

First prove inequality (4.28). The first inequality in (4.28) follows from part (ii) of Lemma 4.12.

Let us prove the second one. By (4.3),  $(9Q) \cap E \neq \emptyset$ . Thus there exists a cube  $K' = Q(x_{K'}, r_{K'})$  such that  $x_{K'} \in 9Q$ .

Let us consider two cases. First suppose that  $x_{K'} \neq x_K$ . Then  $K' \neq K$ . But

$$(9Q) \cap K' \neq \emptyset \quad \text{and} \quad (9Q) \cap K \neq \emptyset$$



so that

$$\text{dist}(K, K') \leq \text{diam}(9Q) = 9 \text{diam } Q.$$

By part (i) of Lemma 3.6,

$$\text{diam } K + \text{diam } K' \leq \text{dist}(K, K')$$

so that

$$\text{diam } K \leq 9 \text{diam } Q.$$

Now suppose that  $x_{K'} = x_K$  so that  $x_K \in 9Q$ . Hence  $\|x_K - x_Q\| \leq 9r_Q$ . On the other hand  $Q \cap (\eta K) = \emptyset$ , see (4.29), so that

$$\|x_K - x_Q\| \geq \eta r_K.$$

Hence  $r_K \leq 9\eta r_Q$ .

We have proved that in the both cases  $\text{diam } K \leq 9\eta \text{diam } Q$  so that the second inequality in (4.28) is satisfied.

Let us prove (4.27). By part (i) of Corollary 4.13,  $Q \subset 5K$  for every  $Q \in J_K$ . Since the cubes of the family  $J_K$  are non-overlapping, this inclusion and the second inequality in (4.28) enable us to estimate the cardinality of  $J_K$ . We have

$$\#J_K \leq |5K| / \min\{|Q| : Q \in J_K\} \leq 5^n |K| / (\eta \text{diam } K / 9)^n = (45/\eta)^n$$

proving (4.27) and the lemma. ■

**Proposition 4.15** *Suppose that the hypothesis of Theorem 3.1 holds with  $\gamma = 2^8 \tau^2$ . Then*

$$\sum_{K \in \mathcal{A}_2} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}} \leq C(n, \tau) \lambda.$$

*Proof.* Following the same scheme of the proof as in Proposition 4.11, without loss of generality we may assume that  $\mathcal{A}_2$  is a finite collection of pairwise disjoint cubes. However, for the family  $\mathcal{A}_2$  we can not define the family  $\mathcal{Q}$  by the same formula as for the family  $\mathcal{A}_1$ , i.e., to put  $\mathcal{Q} := \mathcal{A}_2 \cup \mathcal{K}_E$ . In fact, in this case the cubes of the family  $\mathcal{A}_2$  intersect cubes of  $\mathcal{K}_E$  so that the cubes of  $\mathcal{Q} := \mathcal{A}_2 \cup \mathcal{K}_E$  are not pairwise disjoint.

We modify the definition of  $\mathcal{Q}$  as follows. Let

$$I_K := \sum_{Q \in J_K} \frac{|\tilde{f}_1(a_{\tilde{Q}}) - \tilde{f}_1(a_Q)|^p}{(\text{diam } Q)^{p-n}}$$

so that

$$I := \sum_{K \in \mathcal{A}_2} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}} = \sum_{K \in \mathcal{K}_E} I_K.$$

Let  $H_K \in J_K$  be a cube such that

$$\max_{Q \in J_K} \frac{|\tilde{f}_1(a_{\tilde{Q}}) - \tilde{f}_1(a_Q)|^p}{(\text{diam } Q)^{p-n}} = \frac{|\tilde{f}_1(a_{\tilde{H}_K}) - \tilde{f}_1(a_{H_K})|^p}{(\text{diam } H_K)^{p-n}}.$$

(Of course,  $H_K$  depends on  $f$  as well.) Then, by (4.27),

$$I_K \leq \#J_K \frac{|\tilde{f}_1(a_{\tilde{H}_K}) - \tilde{f}_1(a_{H_K})|^p}{(\text{diam } H_K)^{p-n}} \leq (27/\eta)^n \frac{|\tilde{f}_1(a_{\tilde{H}_K}) - \tilde{f}_1(a_{H_K})|^p}{(\text{diam } H_K)^{p-n}}.$$

Let

$$\widehat{K} := K + 3r_K \vec{e}, \quad \text{where } \vec{e} := (1, 0, \dots, 0).$$

Then  $\widehat{K} \subset 5K \setminus K$ , and  $5K \subset 7\widehat{K}$ . Since  $H_K \in J_K$ , by part (i) of Corollary 4.13,  $H_K \subset 5K$  so that  $H_K \subset 7\widehat{K}$ . Also, by (4.28),

$$(4.30) \quad \text{diam } \widehat{K}/(9\eta) = \text{diam } K/(9\eta) \leq \text{diam } H_K.$$

Now we have

$$I = \sum_{K \in \mathcal{K}_E} I_K \leq C \sum_{K \in \mathcal{K}_E} \frac{|\tilde{f}_1(a_{\tilde{H}_K}) - \tilde{f}_1(a_{H_K})|^p}{(\text{diam } H_K)^{p-n}}$$

so that, by (4.30),

$$(4.31) \quad I \leq C \sum_{K \in \mathcal{K}_E} \frac{|\tilde{f}_1(a_{\tilde{H}_K}) - \tilde{f}_1(a_{H_K})|^p}{(\text{diam } \widehat{K})^{p-n}}$$

with  $C = C(n, \tau)$ .

We introduce a family of cubes

$$\mathcal{Q} := \left( \bigcup_{K \in \mathcal{K}_E} \widehat{K} \right) \cup \mathcal{K}_E.$$

Since  $\widehat{K} \subset 5K \setminus K$  for every  $K \in \mathcal{K}_E$ , and the cubes  $\{5K : K \in \mathcal{K}_E\}$  are pairwise disjoint, see part (i) of Lemma 4.12, the family  $\mathcal{Q}$  consists of pairwise disjoint cubes.

We are in a position to finish the proof of the proposition. Let  $Q \in \mathcal{Q} \setminus \mathcal{K}_E$ , i.e., there exists a cube  $K \in \mathcal{K}_E$  such that  $Q = \widehat{K}$ . We put

$$Q' := K^{(a_{H_K})}, \quad Q'' := K^{(a_{\tilde{H}_Q})}.$$

Then, by definition,

$$\tilde{f}_1(a_{H_K}) = f_{Q'}, \quad \tilde{f}_1(a_{\tilde{H}_Q}) = f_{Q''}.$$

Furthermore, by Lemma 4.9,

$$Q' \cup Q'' \subset (22\tau^2) H_K.$$

But  $H_K \subset 7\widehat{K} = 7Q$  so that

$$Q' \cup Q'' \subset 7(22\tau^2) Q \subset 2^8 \tau^2 Q.$$

If  $Q \in \mathcal{K}_E$ , we put  $Q' = Q'' = Q$ . Hence, by (4.31),

$$(4.32) \quad I \leq C \sum_{Q \in \mathcal{Q}} \frac{|f_{Q'} - f_{Q''}|^p}{(\text{diam } Q)^{p-n}}.$$

Note that the cubes  $Q'$  and  $Q''$  satisfy inequalities (4.25) and (4.26) of Proposition 4.11. Combining these inequalities with (4.32), we obtain

$$I \leq C \sum_{Q \in \mathcal{Q}} \left( \frac{\text{diam } Q' \text{ diam } Q''}{\text{diam } Q} \right)^{p-n} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y),$$

so that, by the assumption (see inequality (3.2)) and in view of Corollary 4.3, we have  $I \leq C\lambda$ . The proposition is completely proved.  $\blacksquare$

This proposition and Proposition 4.15 imply the following

**Corollary 4.16** *The following inequality*

$$\sum_{K \in \mathcal{A}} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}} \leq C(n, \tau)\lambda$$

is satisfied provided the hypothesis of Theorem 3.1 holds with  $\gamma = 2^8\tau^2$ .

*Proof.* Recall that

$$\sum_{K \in \mathcal{A}} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}} \leq C \sum_{K \in \mathcal{A}_1} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}} + C \sum_{K \in \mathcal{A}_2} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}}.$$

It remains to apply Proposition 4.11 to the first sum, and Proposition 4.15 to the second sum in the right hand side of this inequality, and the corollary follows.  $\blacksquare$

Finally, combining this corollary with Corollary 4.8 we obtain the required inequality

$$(4.33) \quad \|\nabla f_1\|_{L_p(\mathbf{R}^n)}^p \leq C\lambda$$

provided  $\gamma = 2^8\tau^2$  and  $C = C(n, \tau)$ .

## 5. Sufficiency: the $L_p(\mathbf{R}^n; \mu)$ -norm of the function $f_2$ .

In this section we prove that

$$(5.1) \quad \|f_2\|_{L_p(\mathbf{R}^n; \mu)}^p \leq C\lambda.$$

**Lemma 5.1** *For every cube  $K \in W_E$  and every  $c \in \mathbf{R}$  the following inequality*

$$\int_K |f_1(x) - c|^p d\mu(x) \leq \left( \sum_{Q \in V_K} |\tilde{f}_1(a_Q) - c| \right)^p \mu(K)$$

holds.

*Proof.* By the extension formula (4.6),

$$\begin{aligned}
\int_K |f_1(x) - c|^p d\mu &= \int_K \left| \sum_{Q \in W_E} \varphi_Q(x) \tilde{f}_1(a_Q) - c \right|^p d\mu \\
&= \int_K \left| \sum_{Q \in W_E} \varphi_Q(x) (\tilde{f}_1(a_Q) - c) \right|^p d\mu \\
&= \int_K \left| \sum \left\{ \varphi_Q(x) (\tilde{f}_1(a_Q) - c) : Q^* \cap K \neq \emptyset, Q \in W_E \right\} \right|^p d\mu \\
&\leq \int_K \left( \sum \left\{ |\varphi_Q(x)| |\tilde{f}_1(a_Q) - c| : Q^* \cap K \neq \emptyset, Q \in W_E \right\} \right)^p d\mu.
\end{aligned}$$

By part (a) of Lemma 4.5 and part (3) of Lemma 4.4,

$$\begin{aligned}
\int_K |f_1(x) - c|^p d\mu &\leq \int_K \left( \sum \left\{ |\tilde{f}_1(a_Q) - c| : Q^* \cap K \neq \emptyset, Q \in W_E \right\} \right)^p d\mu \\
&= \mu(K) \left( \sum \left\{ |\tilde{f}_1(a_Q) - c| : Q \cap K \neq \emptyset, Q \in W_E \right\} \right)^p \\
&= \mu(K) \left( \sum_{Q \in V_K} |\tilde{f}_1(a_Q) - c| \right)^p
\end{aligned}$$

proving the lemma. ■

Let  $K \in \mathcal{K}_E$  and let

$$(5.2) \quad S_K := (\eta K) \bigcup \{Q : Q \in W_E, Q \cap (\eta K) \neq \emptyset\}.$$

(Recall that  $\eta = \frac{1}{2^{1/\tau}}$ , see (4.11).)

**Lemma 5.2** *For every  $K \in \mathcal{K}_E$  the following inequality*

$$\int_{S_K} |f_2|^p d\mu \leq \int_K |f - f_K|^p d\mu$$

*holds.*

*Proof.* First prove that

$$(5.3) \quad f_1(y) = f_K \quad \text{for every } y \in S_K.$$

In fact, since  $K \in \mathcal{K}_E$ , there exists  $x \in E$  such that  $K = K^{(x)}$ . Recall that, by definitions (4.5) and (4.6),

$$f_1(x) = \tilde{f}_1(x) = f_K.$$

Let  $y \in S_K \setminus \{x\}$  so there exists a cube  $H \in W_E$  such that  $y \in H$  and  $H \cap (\eta K) \neq \emptyset$ . By formula (4.6) and part (c) of Lemma 4.5,

$$\begin{aligned} I &:= f_1(y) - \tilde{f}_1(x) = \left( \sum_{Q \in W_E} \varphi_Q(y) \tilde{f}_1(a_Q) \right) - \tilde{f}_1(x) \\ &= \sum_{Q \in W_E} \varphi_Q(y) (\tilde{f}_1(a_Q) - \tilde{f}_1(x)) \\ &= \sum \{ \varphi_Q(y) (\tilde{f}_1(a_Q) - \tilde{f}_1(x)) : Q \in W_E, Q^* \cap H \neq \emptyset \}. \end{aligned}$$

Hence, by part (3) of Lemma 4.4,

$$\begin{aligned} I &= \sum \{ \varphi_Q(y) (\tilde{f}_1(a_Q) - \tilde{f}_1(x)) : Q \in W_E, Q \cap H \neq \emptyset \} \\ &= \sum_{Q \in V_H} \varphi_Q(y) (\tilde{f}_1(a_Q) - \tilde{f}_1(x)). \end{aligned}$$

Since  $H \cap (\eta K) \neq \emptyset$ , by Lemma 4.7,  $a_Q = x$  for every  $Q \in V_H$  so that

$$I := f_1(y) - \tilde{f}_1(x) = f_1(y) - f_K = 0,$$

and (5.3) follows.

Furthermore, since  $Q \in V_Q$ , by Lemma 4.7,  $Q \subset K^{(x)} = K$  for every  $Q \in W_E$  such that  $Q \cap (\eta K) \neq \emptyset$ . Hence  $S_K \subset K$ .

Finally,

$$\int_{S_K} |f_2|^p d\mu = \int_{S_K} |f(y) - f_1(y)|^p d\mu(y) = \int_{S_K} |f(y) - f_K|^p d\mu(y) \leq \int_K |f - f_K|^p d\mu$$

proving the lemma. ■

Let  $\mathcal{A}$  be the family of cubes defined by (4.17). Thus  $K \in \mathcal{A} \Leftrightarrow K \in W_E$  and  $K \cap (\eta T) = \emptyset$  for every cube  $T \in \mathcal{K}_E$ . Our next goal is to prove that

$$(5.4) \quad \sum_{K \in \mathcal{A}} \int_K |f_2|^p d\mu \leq C \lambda.$$

We have

$$\sum_{K \in \mathcal{A}} \int_K |f_2|^p d\mu = \sum_{K \in \mathcal{A}} \int_K |f - f_1|^p d\mu$$

so that

$$(5.5) \quad \sum_{K \in \mathcal{A}} \int_K |f_2|^p d\mu \leq 2^p \left\{ \sum_{K \in \mathcal{A}} \int_K |f_1 - \tilde{f}_1(a_K)|^p d\mu + \sum_{K \in \mathcal{A}} \int_K |f - \tilde{f}_1(a_K)|^p d\mu \right\}.$$

**Proposition 5.3** *Suppose that the hypothesis of Theorem 3.1 holds with  $\gamma = 2^8\tau^2$ . Then*

$$\sum_{K \in \mathcal{A}} \int_K |f_1 - \tilde{f}_1(a_K)|^p d\mu \leq C(n, p, \tau) \lambda.$$

*Proof.* By Lemma 5.1,

$$\int_K |f_1(x) - \tilde{f}_1(a_K)|^p d\mu(x) \leq \left( \sum_{Q \in V_K} |\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)| \right)^p \mu(K)$$

Recall that  $V_K = \{Q \in W_E : Q \cap K \neq \emptyset\}$ , and  $\#V_K \leq N(n)$ , see part (2) of Lemma 4.4. We also recall that by  $\tilde{K} \in V_K$  we denote a cube satisfying (4.9). Hence,

$$\begin{aligned} \left( \sum_{Q \in V_K} |\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)| \right)^p \mu(K) &\leq (\#V_K)^p \mu(K) |\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p \\ &\leq C(n) \mu(K) |\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p. \end{aligned}$$

We obtain

$$\int_K |f_1(x) - \tilde{f}_1(a_K)|^p d\mu(x) \leq C(n) \mu(K) |\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p.$$

Since  $K \in W_E$ , by Lemma 4.2,  $\mu(K) \leq 2^{15p}(\text{diam } K)^{n-p}$ , so that

$$\int_K |f_1(x) - \tilde{f}_1(a_K)|^p d\mu(x) \leq C(n, p) \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}}.$$

Hence,

$$(5.6) \quad \sum_{K \in \mathcal{A}} \int_K |f_1 - \tilde{f}_1(a_K)|^p d\mu \leq C(n, p) \sum_{K \in \mathcal{A}} \frac{|\tilde{f}_1(a_{\tilde{K}}) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}}$$

so that, by Corollary 4.16,

$$\sum_{K \in \mathcal{A}} \int_K |f_1 - \tilde{f}_1(a_K)|^p d\mu \leq C(n, p, \tau) \lambda$$

provided the hypothesis of Theorem 3.1 holds with  $\gamma = 2^8\tau^2$ . ■

Let

$$(5.7) \quad I_2 := \sum_{K \in \mathcal{A}} \int_K |f - \tilde{f}_1(a_K)|^p d\mu + \sum_{K \in \mathcal{K}_E} \int_K |f - f_K|^p d\mu.$$

We define a collection of cubes

$$(5.8) \quad \mathcal{Q} := \mathcal{A} \cup \mathcal{K}_E.$$

As we have noted in the proof of Proposition 4.11, see a remark after (4.22), the cubes of  $\mathcal{Q}$  are pairwise disjoint. Furthermore, these cubes satisfy inequality (4.2) so that the conditions (3.3) of Theorem 3.1 hold.

Let  $Q \in \mathcal{Q}$ . We define two cubes  $Q', Q'' \in \mathcal{Q}$ ,  $Q' \cup Q'' \subset \gamma Q$ , as follows. If  $Q = K \in \mathcal{A}$ , we put

$$(5.9) \quad Q' := Q = K, \quad Q'' := K^{(a_K)},$$

(c.f. definition (4.23)). Then, by definition (4.5),

$$\tilde{f}_1(a_K) := f_{Q''}.$$

Furthermore, by Lemma 4.9,  $Q' \cup Q'' \subset \gamma Q$  with  $\gamma = 22\tau^2$ .

If  $Q \in \mathcal{K}_E$ , i.e.,  $Q = K^{(x)}$  for some  $x \in E$ , we put

$$(5.10) \quad Q' = Q'' := Q.$$

Thus in this case

$$Q' \cup Q'' \subset \gamma Q \quad \text{with} \quad \gamma = 1.$$

**Proposition 5.4** *We have*

$$I_2 \leq \sum_{Q \in \mathcal{Q}} \left( \frac{\text{diam } Q' \text{ diam } Q''}{\text{diam } Q} \right)^{p-n} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y).$$

*Proof.* The proof of this proposition is very similar to that of Proposition 4.11. First, as in that proof, without loss of generality we may assume that  $\mathcal{A}$  is a finite collection of pairwise disjoint cubes.

Note that  $Q'' \in \mathcal{K}_E$  for every  $Q \in \mathcal{Q}$  so that, by Corollary 3.9, see (3.28),

$$(5.11) \quad (\text{diam } Q'')^{n-p} \leq 2^{n-p} \mu(Q'') \leq \mu(Q'').$$

Also, since  $Q' = Q$  for every  $Q \in \mathcal{Q}$ , we have

$$(5.12) \quad \text{diam } Q' = \text{diam } Q, \quad Q \in \mathcal{Q}.$$

Hence,

$$I_2 := \sum_{K \in \mathcal{A}} \int_K |f - \tilde{f}_1(a_K)|^p d\mu + \sum_{K \in \mathcal{K}_E} \int_K |f - f_K|^p d\mu = \sum_{Q \in \mathcal{Q}} \int_{Q'} |f - f_{Q''}|^p d\mu.$$

But

$$\int_{Q'} |f - f_{Q''}|^p d\mu \leq \frac{1}{\mu(Q'')} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)$$

so that, by (5.11),

$$\int_{Q'} |f - f_{Q''}|^p d\mu \leq (\text{diam } Q'')^{p-n} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y).$$

Hence, by (5.12),

$$\int_{Q'} |f - f_{Q''}|^p d\mu \leq \left( \frac{\text{diam } Q' \text{ diam } Q''}{\text{diam } Q} \right)^{p-n} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)$$

so that

$$I_2 = \sum_{Q \in \mathcal{Q}} \int_{Q'} |f - f_{Q''}|^p d\mu \leq \sum_{Q \in \mathcal{Q}} \left( \frac{\text{diam } Q' \text{ diam } Q''}{\text{diam } Q} \right)^{p-n} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)$$

proving the proposition. ■

**Corollary 5.5** *Suppose that the hypothesis of Theorem 3.1 holds with  $\gamma = 22\tau^2$ . Then*

$$I_2 = \sum_{K \in \mathcal{A}} \int_K |f - \tilde{f}_1(a_K)|^p d\mu + \sum_{K \in \mathcal{K}_E} \int_K |f - f_K|^p d\mu \leq \lambda.$$

*Proof.* Note that the family  $\mathcal{Q}$  and the cubes  $Q', Q''$  satisfy all the conditions of Theorem 3.1. Then, by the theorem's hypothesis,  $I_2 \leq \lambda$ , proving the corollary. ■

Combining this corollary with Proposition 5.3 we obtain the required inequality (5.4). In fact,

$$\begin{aligned} \sum_{K \in \mathcal{A}} \int_K |f_2|^p d\mu &\leq 2^p \left\{ \sum_{K \in \mathcal{A}} \int_K |f_1 - \tilde{f}_1(a_K)|^p d\mu + \sum_{K \in \mathcal{A}} \int_K |f - \tilde{f}_1(a_K)|^p d\mu \right\} \\ &\leq C(n, p, \tau) \lambda \end{aligned}$$

provided the hypothesis of Theorem 3.1 hold with  $\gamma = 2^8 \tau^2$ .

Note that Proposition 5.5 implies the following

**Corollary 5.6** *The following inequality*

$$\sum_{K \in \mathcal{K}_E} \int_{S_K} |f_2|^p d\mu \leq \lambda$$

*holds.*

*Proof.* By Corollary 5.5,

$$\sum_{K \in \mathcal{K}_E} \int_K |f - f_K|^p d\mu \leq \lambda$$

so that, by Lemma 5.2,

$$\sum_{K \in \mathcal{K}_E} \int_{S_K} |f_2|^p d\mu \leq \sum_{K \in \mathcal{K}_E} \int_K |f - f_K|^p d\mu \leq \lambda$$

proving the corollary. ■



To finish the proof of Theorem 3.1 it remains to note that the collection of subsets  $\{S_K : K \in \mathcal{K}_E\}$ , see (5.2), and the family of cubes  $\mathcal{A}$ , see (4.17), cover  $\mathbf{R}^n$ . Hence

$$(5.13) \quad \int_{\mathbf{R}^n} |f_2|^p d\mu \leq \sum_{K \in \mathcal{K}_E} \int_{S_K} |f_2|^p d\mu + \sum_{K \in \mathcal{A}} \int_K |f_2|^p d\mu.$$

Combining this inequality with Corollary 5.6 and Corollary 5.4 we obtain the required inequality (5.1) provided  $\gamma = 2^8 \tau^2$  and  $C = C(n, \tau)$ .

In turn, combining (5.1) with inequality (4.33), we finally obtain the required estimate (4.7).

Theorem 3.1 is completely proved. ■

## 6. Refinements of the criterion for the norm in $L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$ .

### 6.1 A refinement of Theorem 1.3.

**Theorem 6.1** *Let  $n < p < \infty$  and let  $\mu$  be a non-trivial non-negative Borel measure on  $\mathbf{R}^n$ . There exist constants  $\gamma = \gamma(n) > 0$  and  $N = N(n) \in \mathbf{N}$ , a family  $\mathcal{Q}$  consisting of pairwise disjoint cubes and a family  $\tilde{\mathcal{Q}}$  of cubes in  $\mathbf{R}^n$  with covering multiplicity  $M(\tilde{\mathcal{Q}}) \leq N$ , mappings*

$$(6.1) \quad \mathcal{Q} \ni Q \mapsto Q' \in \tilde{\mathcal{Q}} \quad \text{and} \quad \mathcal{Q} \ni Q \mapsto Q'' \in \tilde{\mathcal{Q}}$$

*satisfying the condition*

$$(6.2) \quad Q' \cup Q'' \subset \gamma Q \quad \text{for all} \quad Q \in \mathcal{Q},$$

*such that for every function  $f \in L_{p,loc}(\mathbf{R}^n; \mu)$  the following equivalence*

$$(6.3) \quad \|f\|_{\Sigma} \sim \left( \sum_{Q \in \mathcal{Q}} \frac{(\text{diam } Q)^{n-p} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)}{\{(\text{diam } Q')^{n-p} + \mu(Q')\} \{(\text{diam } Q'')^{n-p} + \mu(Q'')\}} \right)^{\frac{1}{p}}$$

*holds. The constants of this equivalence depend only on  $n$  and  $p$ .*

*Proof.* We let  $I(f; \mathcal{Q})$  denote the quantity from the right-hand side of the equivalence (6.3). Then inequality

$$I(f; \mathcal{Q}) \leq C(n, p) \|f\|_{\Sigma}$$

follows from Proposition 2.3 where one can put  $\mathcal{S} = \mathcal{Q}$ . In fact, it can be easily seen that  $I(f; \mathcal{Q})$  does not exceed the quantity in the left-hand side of inequality (2.4).

Prove that for certain families  $\mathcal{Q}$  and  $\tilde{\mathcal{Q}}$  and mappings from (6.1) with condition (6.2) each depending only on  $p, n$ , and the measure  $\mu$ , we have

$$\|f\|_{\Sigma} \leq C(n, p) I(f; \mathcal{Q})$$

provided  $f \in L_{p,loc}(\mathbf{R}^n; \mu)$  is an arbitrary function.

We construct these objects using the method of proof of the sufficiency part of Theorem 1.3.

Let  $E$  be the set constructing in Proposition 3.5 and let  $\mathcal{K}_E$  be the family of cubes defined by (3.24).

Let  $f_1$  be the function defined by formula (4.6) and let  $f_2 = f - f_1$ . As before given a cube  $K \in W_E$  we put

$$V_K := \{Q \in W_E : Q \cap K \neq \emptyset\}.$$

Then, by (4.8),

$$\|f_1\|_{L_p^1(\mathbf{R}^n)}^p \leq C(n) \sum_{K \in W_E} \sum_{Q \in V_K} \frac{|\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}}.$$

Recall that, by (4.5),

$$(6.4) \quad \tilde{f}_1(x) := f_Q = \frac{1}{\mu(Q)} \int_Q f \, d\mu \quad \text{for every } x \in E,$$

where  $Q = K^{(x)} = Q(x, R(x))$  is the unique cube from  $\mathcal{K}_E$  with center at the point  $x \in E$ .

Also recall that, by Lemma 4.7,

$$(6.5) \quad \|f_1\|_{L_p^1(\mathbf{R}^n)}^p \leq C(n) I_1$$

where

$$(6.6) \quad I_1 := \sum_{K \in \mathcal{A}} \sum_{Q \in V_K} \frac{|\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}}.$$

Here  $\mathcal{A} \subset W_E$  is the family of cubes defined by (4.17).

Fix a cube  $K \in \mathcal{A}$ . Let  $Q \in V_K$ . Recall that, by (4.4),  $a_Q \in \tau Q$ . Since  $Q \cap K \neq \emptyset$  and  $\text{diam } Q \leq 4 \text{diam } K$ , we have  $Q \subset 8K$ . Hence  $a_Q \in 8\tau K$ .

Since  $K \in \mathcal{A}$ , we have  $K \cap K^{(a_Q)} = \emptyset$ . These properties of  $a_Q$  and  $K^{(a_Q)}$  easily imply that

$$(6.7) \quad K^{(a_Q)} \subset \gamma_1 K \quad \text{for every } Q \in V_K$$

with  $\gamma_1 = 156\tau$ .

Let us divide  $K$  into  $2^n$  equal cubes of diameter  $\frac{1}{2} \text{diam } K$ . Let us fix one of these cubes and denote that cube by  $\tilde{K}$ . Let

$$(6.8) \quad T_K := \frac{1}{2} \tilde{K}.$$

Let  $V_K \setminus \{K\} = \{Q_1, Q_2, \dots, Q_m\}$ . We know that  $m = m(K) \leq C(n)$ . Obviously there exists a family of *pairwise disjoint* equal cubes

$$Y_K := \{\tilde{Q}_1, \tilde{Q}_2, \dots, \tilde{Q}_m\}$$

such that  $\tilde{Q}_i \subset T_K$  and

$$\text{diam } K \leq C(n) \text{diam } \tilde{Q}_i, \quad \text{for every } i = 1, \dots, m.$$

Here  $C = C(n)$  is a constant depending only on  $n$ . Then, clearly,  $K \subset \gamma_2 \tilde{Q}_i$  for every  $1 \leq i \leq m$  with certain  $\gamma_2 = \gamma_2(n)$  so that, by (6.7),

$$(6.9) \quad K^{(a_{Q_i})}, K^{(a_K)} \subset \gamma_3(n) \tilde{Q}_i \quad \text{for every } i = 1, \dots, m.$$

Let  $Q \in Y_K$ ; thus  $Q = \tilde{Q}_i$  for some  $1 \leq i \leq m$ . We assign to  $Q$  cubes  $Q', Q'' \in \mathcal{K}_E$  as follows:

$$(6.10) \quad Q' := K^{(a_{Q_i})} \quad \text{and} \quad Q'' := K^{(a_K)}.$$

Thus  $Q', Q'' \in \mathcal{K}_E$ .

Then, by (6.9),  $Q', Q'' \subset \gamma_3 Q$ .

Furthermore, since  $\text{diam } Q \sim \text{diam } K$  for each  $Q \in Y_K$ , by (6.4),

$$\sum_{Q \in V_K} \frac{|\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}} \sim \sum_{Q \in \mathcal{Q}_K} \frac{|f_{Q'} - f_{Q''}|^p}{(\text{diam } Q)^{p-n}}.$$

Let  $\mathcal{Q}_1 = \cup\{Y_K : K \in \mathcal{A}\}$ . We obtain

$$I_1 \sim \sum_{Q \in \mathcal{Q}_1} \frac{|f_{Q'} - f_{Q''}|^p}{(\text{diam } Q)^{p-n}}.$$

But

$$|f_{Q'} - f_{Q''}|^p \leq \frac{1}{\mu(Q')\mu(Q'')} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x)d\mu(y)$$

so that

$$I_1 \leq C(n) \sum_{Q \in \mathcal{Q}_1} \frac{(\text{diam } Q)^{p-n}}{\mu(Q')\mu(Q'')} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x)d\mu(y).$$

Since  $Q', Q'' \in \mathcal{K}_E$ , by (3.28),  $\mu(Q') \sim (\text{diam } Q')^{n-p}$  and  $\mu(Q'') \sim (\text{diam } Q'')^{n-p}$  so that

$$(6.11) \quad I_1 \leq C(n) \sum_{Q \in \mathcal{Q}_1} \frac{(\text{diam } Q)^{n-p} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x)d\mu(y)}{\{(\text{diam } Q')^{n-p} + \mu(Q')\}\{(\text{diam } Q'')^{n-p} + \mu(Q'')\}}.$$

Combining this inequality with (6.5) we obtain

$$(6.12) \quad \|f_1\|_{L_p^1(\mathbf{R}^n)}^p \leq C(n) \sum_{Q \in \mathcal{Q}_1} \frac{(\text{diam } Q)^{n-p} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x)d\mu(y)}{\{(\text{diam } Q')^{n-p} + \mu(Q')\}\{(\text{diam } Q'')^{n-p} + \mu(Q'')\}}.$$

Let us estimate  $\|f_2\|_{L_p(\mathbf{R}^n; \mu)}$  using the scheme of the proof of the sufficiency part of Theorem 1.3 given in Section 5. We will also use the settings of this section.

By (5.6),

$$\sum_{K \in \mathcal{A}} \int_K |f_1 - \tilde{f}_1(a_K)|^p d\mu \leq C(n, p) \sum_{K \in \mathcal{A}} \sum_{Q \in V_K} \frac{|\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}}$$

so that, by (6.6),

$$(6.13) \quad \sum_{K \in \mathcal{A}} \int_K |f_1 - \tilde{f}_1(a_K)|^p d\mu \leq C(n, p) I_1.$$

In turn, by (5.13),

$$\int_{\mathbf{R}^n} |f_2|^p d\mu \leq \sum_{K \in \mathcal{K}_E} \int_{S_K} |f_2|^p d\mu + \sum_{K \in \mathcal{A}} \int_K |f_2|^p d\mu.$$

Combining this inequality with (5.5), Lemma 5.2 and (6.13), we obtain

$$(6.14) \quad \int_{\mathbf{R}^n} |f_2|^p d\mu \leq C\{I_1 + I_2\}.$$

Recall that  $I_2$  is defined by (5.7).

Let  $\mathcal{Q}$  be the family of cubes defined by (5.8). Let us slightly modify this family as follows. Recall for each cube  $K \in W_E$  we have introduced a cube  $\tilde{K}$  as one of the cubes from partition of  $K$  into the family of  $2^n$  equal cubes. See (6.8).

Let us introduce another cube from this partition and denote this cube by  $\hat{K}$ . Thus  $\text{diam } \hat{K} = \frac{1}{2} \text{diam } K$  and  $\tilde{K} \neq \hat{K}$  for every cube  $K \in W_E$ .

Let

$$\hat{\mathcal{A}} = \{\tfrac{1}{2}\hat{Q} : Q \in \mathcal{A}\}$$

and let

$$\mathcal{Q}_2 := \hat{\mathcal{A}} \cup \mathcal{K}_E.$$

In other words we replace in definition (5.8) the family  $\mathcal{A}$  with the family  $\hat{\mathcal{A}}$ . Since  $\text{diam } \hat{Q} \sim \text{diam } Q$  and  $Q \subset 3\hat{Q}$  the result of Proposition 5.4 remains true after such a modification, i.e.,

$$I_2 \leq \sum_{Q \in \mathcal{Q}_2} \left( \frac{\text{diam } Q' \text{diam } Q''}{\text{diam } Q} \right)^{p-n} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y).$$

Note that for each cube  $Q \in \mathcal{Q}_2$  we have  $\mu(Q') \leq C(\text{diam } Q')^{n-p}$  and the same is true for  $Q''$ . Hence

$$(6.15) \quad I_2 \leq C \sum_{Q \in \mathcal{Q}_2} \frac{(\text{diam } Q)^{n-p} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)}{\{(\text{diam } Q')^{n-p} + \mu(Q')\} \{(\text{diam } Q'')^{n-p} + \mu(Q'')\}}.$$

Also we remark that by definition  $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \emptyset$ , i.e., the family

$$\mathcal{Q} := \mathcal{Q}_1 \cup \mathcal{Q}_2$$

consists of *pairwise disjoint* cubes. Furthermore, given  $Q \in \mathcal{Q}$  the cubes  $Q'$  and  $Q''$  belong to the family

$$\tilde{\mathcal{Q}} := \mathcal{A} \cup \mathcal{K}_E$$

See (6.10), (5.9) and (5.10). Since the cubes of the family  $\mathcal{K}_E$  are pairwise disjoint and  $\mathcal{A} \subset W_E$ , covering multiplicity of the family  $\tilde{\mathcal{Q}}$  satisfies the following inequality

$$M(\tilde{\mathcal{Q}}) = M(\mathcal{A} \cup \mathcal{K}_E) \leq M(\mathcal{A}) + M(\mathcal{K}_E) \leq M(W_E) + 1 \leq N(n).$$

Now combining inequality (6.15) with inequalities (6.14) and (6.11), we obtain

$$\|f_2\|_{L_p(\mathbf{R}^n; \mu)}^p \leq C \sum_{Q \in \mathcal{Q}} \frac{(\text{diam } Q)^{n-p} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)}{\{(\text{diam } Q')^{n-p} + \mu(Q')\} \{(\text{diam } Q'')^{n-p} + \mu(Q'')\}}.$$

Finally, this inequality and (6.12) imply that

$$\begin{aligned} \|f\|_{\Sigma}^p &\leq 2^p \{ \|f_1\|_{L_p^1(\mathbf{R}^n)}^p + \|f_2\|_{L_p(\mathbf{R}^n; \mu)}^p \} \\ &\leq C \sum_{Q \in \mathcal{Q}} \frac{(\text{diam } Q)^{n-p} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)}{\{(\text{diam } Q')^{n-p} + \mu(Q')\} \{(\text{diam } Q'')^{n-p} + \mu(Q'')\}} \end{aligned}$$

proving the theorem. ■

**6.2. Lacunae of Whitney's cubes.** In the next subsection we present another refinement of Theorem 1.3. We obtain this refinement with the help of a modification of the classical Whitney extension method which we described and used at the beginning of Section 4. See formula (4.6).

As we have noted in Section 1 the main idea of this approach is to use certain *families* of Whitney's cubes rather than to treat each Whitney cube separately. We call these families of Whitney cubes *lacunae*.

In this subsection we present main definitions and main properties of lacunae. For the proof of these properties we refer the reader to paper [12], Sections 4-5.

Let  $E$  be a closed subset of  $\mathbf{R}^n$  and let  $W_E$  be a Whitney decomposition of its complement  $\mathbf{R}^n \setminus E$ , see Theorem 4.1 and Lemma 4.4. As we have already noted in Section 4, see (4.3),

$$(6.16) \quad (9Q) \cap E \neq \emptyset \quad \text{for every } Q \in W_E.$$

By  $LW_E$  we denote a subfamily of Whitney cubes satisfying the following condition:

$$(6.17) \quad (10Q) \cap E = (9Q) \cap E.$$

Then we introduce a binary relation  $\sim$  on  $LW_E$ : for every  $Q_1, Q_2 \in LW_E$

$$Q_1 \sim Q_2 \iff (10Q_1) \cap E = (10Q_2) \cap E.$$

It can be easily seen that  $\sim$  satisfies the axioms of equivalence relations, i.e., it is reflexive, symmetric and transitive. Given a cube  $Q \in LW_E$  by

$$[Q] := \{K \in LW_E : K \sim Q\}$$

we denote the equivalence class of  $Q$ . We refer to this equivalence class as a *true lacuna* with respect to the set  $E$ .

Let

$$\tilde{\mathcal{L}}_E = LW_E \setminus \sim = \{[Q] : Q \in LW_E\}$$

be the corresponding quotient set of  $LW_E$  by  $\sim$ , i.e., the set of all possible equivalence classes (lacunae) of  $LW_E$  by  $\sim$ .

Thus for every pair of Whitney cubes  $Q_1, Q_2 \in W_E$  which belong to a true lacuna  $L \in \tilde{\mathcal{L}}_E$  we have

$$(6.18) \quad (10Q_1) \cap E = (90Q_1) \cap E = (10Q_2) \cap E = (90Q_2) \cap E.$$

By  $V_L$  we denote the associated set of the lacuna  $L$

$$(6.19) \quad V_L := (90Q) \cap E.$$

Here  $Q$  is an arbitrary cube from  $L$ . By (6.18), any choice of a cube  $Q \in L$  provides the same set  $V_L$  so that  $V_L$  is well-defined. Also note that for each cube  $Q$  which belong to a true lacuna  $L$  we have  $V_L = (10Q) \cap E$ .

We extend the family  $\tilde{\mathcal{L}}_E$  of true lacunae to a family  $\mathcal{L}_E$  of *all lacunae* in the following way. Suppose that  $Q \in W_E \setminus LW_E$ , see (6.17), i.e.,

$$(6.20) \quad (10Q) \cap E \neq (90Q) \cap E.$$

In this case to the cube  $Q$  we assign a lacuna  $L := \{Q\}$  consisting of a unique cube - the cube  $Q$ . We also put  $V_L := (90Q) \cap E$  as in (6.19).

We refer to such a lacuna  $L := \{Q\}$  as an *elementary lacuna* with respect to the set  $E$ . By  $\hat{\mathcal{L}}_E$  we denote the family of all elementary lacunae with respect to  $E$ :

$$\hat{\mathcal{L}}_E := \{L = \{Q\} : Q \in W_E \setminus LW_E\}$$

We note that property (6.20) implies the existence of a point

$$a \in (E \setminus (10Q)) \cap (90Q).$$

On the other hand, by (6.16), there exists a point

$$b \in (9Q) \cap E.$$

Hence

$$\|a - b\| \geq r_Q = (1/2) \text{diam } Q$$

so that

$$\text{diam } V_L = \text{diam}((90Q) \cap E) \geq \frac{1}{2} \text{diam } Q$$

provided

$$L = \{Q\} \in \hat{\mathcal{L}}_E$$

is an elementary lacuna.

Finally, by  $\mathcal{L}_E$  we denote the family of all lacunae with respect to  $E$ :

$$\mathcal{L}_E = \tilde{\mathcal{L}}_E \cup \hat{\mathcal{L}}_E.$$

We turn to description of main properties of lacunae. Recall that the detailed proofs of these properties one can find in [12], Sections 4-5.

**Proposition 6.2** *Let  $L \in \mathcal{L}_E$  be a lacuna. If  $\text{diam } V_L > 0$ , then there exists a cube  $Q_L \in L$  such that*

$$\text{diam } Q_L = \min\{\text{diam } Q : Q \in L\}.$$

*Furthermore,*

$$\frac{1}{90} \text{diam } V_L \leq \text{diam } Q_L \leq \gamma \text{diam } V_L$$

*where  $\gamma$  is an absolute constant.*

Given a lacuna  $L \in \mathcal{L}_E$  we let  $U_L$  denote the union of all cubes which belong to the lacuna:

$$U_L := \cup\{Q : Q \in L\}.$$

By  $\text{diam } L$  we denote the diameter of the set  $U_L$ :

$$\text{diam } L := \text{diam } U_L = \sup\{\|a - b\| : a, b \in U_L\}.$$

We say that  $L$  is bounded if  $\text{diam } L < \infty$ . If  $\text{diam } L = \infty$  we say that  $L$  is an unbounded lacuna.

**Proposition 6.3** *(i). For every lacuna  $L \in \mathcal{L}_E$*

$$\text{diam } L \sim \sup\{\text{diam } Q : Q \in L\} \sim \text{dist}(V_L, E \setminus V_L)$$

*with absolute constants in the equivalences;*

*(ii). If  $E$  is an unbounded set then every lacuna  $L \in \mathcal{L}_E$  is bounded;*

*(iii). If  $E$  is bounded, there exists the unique unbounded lacuna  $L^{\max} \in \mathcal{L}_E$ . The lacuna  $L^{\max}$  is a true lacuna for which  $V_{L^{\max}} = E$ .*

**Proposition 6.4** *Let  $L \in \mathcal{L}_E$  be a bounded lacuna. Then there exists a cube  $Q^{(L)} \in L$  such that*

$$\text{diam } Q^{(L)} = \max\{\text{diam } K : K \in L\}.$$

*Furthermore,*

$$(6.21) \quad \text{diam } Q^{(L)} \sim \text{diam } L \sim \text{dist}(V_L, E \setminus V_L),$$

*and*

$$V_L \cup U_L \subset \gamma Q^{(L)}.$$

*Here the constant  $\gamma$  and constants in the equivalences of (6.21) are absolute .*

**Proposition 6.5** *Let  $L \in \mathcal{L}_E$  be a lacuna and let  $Q \in L$ . Suppose that there exist a lacuna  $L' \in \mathcal{L}_E$ ,  $L \neq L'$ , and a cube  $Q' \in L'$  such that  $Q \cap Q' \neq \emptyset$ . Then:*

*(i). If  $L$  is a true lacuna, then  $L'$  is an elementary lacuna, i.e.,  $L' \in \hat{\mathcal{L}}_E = \mathcal{L}_E \setminus \tilde{\mathcal{L}}_E$ ;*

*(ii). Either*

$$\text{diam } Q \sim \text{diam } V_L \sim \text{diam } Q_L$$

*or*

$$\text{diam } Q \sim \text{dist}(V_L, E \setminus V_L) \sim \text{diam } Q^{(L)}$$

*with absolute constants in the equivalences.*

**Proposition 6.6** *Let  $L \in \mathcal{L}_E$  be a lacuna and let*

$$\mathcal{I}_L := \{K \in W_E \setminus L : \exists Q \in L \text{ such that } K \cap Q \neq \emptyset\}.$$

*Then  $\#\mathcal{I}_L \leq \gamma(n)$ .*

One of the main ingredient of the lacunary approach is a mapping  $\mathcal{L}_E \ni L \mapsto \mathcal{PR}(L) \in E$  whose properties are described by the following

**Proposition 6.7** *There exist an absolute constant  $\gamma > 0$  and a mapping*

$$\mathcal{L}_E \ni L \longrightarrow \mathcal{PR}(L) \in E$$

*such that:*

*(i). For every lacuna  $L \in \mathcal{L}_E$  we have*

$$(6.22) \quad \mathcal{PR}(L) \in (\gamma Q_L) \cap E ;$$

*(ii). For every  $a \in E$*

$$\#\{L \in \mathcal{L}_E : \mathcal{PR}(L) = a\} \leq C(n).$$

We refer to the mapping  $\mathcal{PR}$  as a “projection” of  $\mathcal{L}_E$  into the set  $E$ .

Let  $L \in \mathcal{L}_E$  be a lacuna. Recall that

$$U_L = \cup\{Q : Q \in L\}.$$

**Definition 6.8** Let  $L, L' \in \mathcal{L}_E$  be lacunae. We say that  $L$  and  $L'$  are *contacting lacunae* if  $U_L \cap U_{L'} \neq \emptyset$ . In this case we write  $L \leftrightarrow L'$ .

Thus  $L \leftrightarrow L'$  whenever there exist cubes  $Q \in L$  and  $Q' \in L'$  such that  $Q \cap Q' \neq \emptyset$ . We refer to the pair of such cubes as *contacting cubes*. Let us present several properties of contacting lacunae and contacting cubes.

**Proposition 6.9** *(i). Every lacuna  $L \in \mathcal{L}_E$  contacts with at most  $C(n)$  lacunae, i.e.,*

$$\#\{L' \in \mathcal{L}_E : L' \leftrightarrow L\} \leq C(n);$$

*(ii). Every true lacuna contacts only with elementary lacunae.*

**Proposition 6.10** *Let  $L \in \mathcal{L}_E$  be a lacuna and let  $Q \in L$  be a contacting cube. (I.e., there exist a lacuna  $L' \in \mathcal{L}_E$  and a cube  $Q' \in L'$  such that  $Q \cap Q' \neq \emptyset$ .) Then either*

$$\text{diam } Q \sim \text{diam } V_L \sim \min\{\text{diam } K : K \in L\} = \text{diam } Q_L$$

*or*

$$\text{diam } Q \sim \text{dist}(V_L, E \setminus V_L) \sim \max\{\text{diam } K : K \in L\} = \text{diam } Q^{(L)}$$

*with absolute constants in the equivalences.*



### 6.3. A lacunary refinement of the main result.

The next refinement of Theorem 1.3 is motivated by its important applications to characterization of the restrictions of Sobolev functions to closed subsets in  $\mathbf{R}^2$ . See [12].

**Theorem 6.11** *Let  $\mu$  be a non-trivial non-negative Borel measure on  $\mathbf{R}^n$ ,  $n < p < \infty$ , and let*

$$\Sigma = L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu).$$

*There exist families of closed sets  $\{G_1, G_2, \dots\}$  and  $\{H_1, H_2, \dots\}$  with covering multiplicity  $M(\{G_i\}), M(\{H_i\}) \leq C(n)$ , and a family  $\{\lambda_1, \lambda_2, \dots\}$  of positive numbers such that for every function  $f \in L_{1,loc}(\mathbf{R}^n; \mu)$  the following equivalence*

$$\|f\|_{\Sigma}^p \sim \sum_{i=1}^{\infty} \lambda_i \iint_{G_i \times H_i} |f(x) - f(y)|^p d\mu(x) d\mu(y)$$

*holds. The constants of this equivalence depend only on  $n$  and  $p$ .*

*Proof.* We follow the scheme of the proof of Theorem 6.1. Let  $L \in \mathcal{L}_E$  be a lacuna. For the sake of brevity we put

$$A_L := \mathcal{PR}(L)$$

where  $\mathcal{PR}$  denotes the “projection” of  $L$  into  $E$ , see Proposition 6.7.

We modify the Whitney extension formula as follows: We put

$$a_Q = A_L \quad \text{for all } Q \in L.$$

Cf. (4.4).

Note that, by (6.22),

$$A_L \in (\gamma Q_L) \cap E$$

where  $Q_L$  is a cube in  $L$  of minimal diameter. Since for every  $Q \in L$

$$(90Q) \cap E = (90Q_L) \cap E,$$

we have  $(90Q) \cap (90Q_L) \neq \emptyset$ . But  $\text{diam } Q_L \leq \text{diam } Q$  so that

$$Q_L \subset \gamma_1 Q \quad \text{for every } Q \in L$$

with some absolute  $\gamma_1 > 0$ . Hence

$$(6.23) \quad a_Q = A_L \subset (\gamma_2 Q) \cap E \quad \text{for every } Q \in L.$$

This shows that we can construct the component  $f_1$  using the extension formula (4.6). Then the functions  $f_1$  and  $f_2 = f - f_1$  will provide an almost optimal decomposition of  $f$ , i.e.,

$$\|f_1\|_{L_p^1(\mathbf{R}^n)} + \|f_2\|_{L_p(\mathbf{R}^n; \mu)} \sim \|f\|_{\Sigma}.$$

Let us construct the required families  $\{G_i\}$  and  $\{H_i\}$  using the approach suggested in the proof of Theorem 6.1.

We begin with the estimate of the quantity  $\|f_1\|_{L_p^1(\mathbf{R}^n)}$ . First we modify the estimate (6.5) and definition (6.6).

Let  $L \in \mathcal{L}_E$ . We note that for every  $K \in L$  and every  $Q \in V_K \cap L$  we have  $a_Q = a_K = A_L$  so that

$$|\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)| = 0.$$

Let

$$T(L) := \{K \in L : \exists Q \in W_E \setminus L, Q \cap K \neq \emptyset\}.$$

Thus  $T(L)$  is the family of contacting cubes of  $L$ . By Proposition 6.6 and part (2) of Lemma 4.4,

$$(6.24) \quad \#T(L) \leq C(n), \quad L \in \mathcal{L}_E.$$

These observations enable us to modify inequalities (6.5) and (6.6) as follows:

$$\|f_1\|_{L_p^1(\mathbf{R}^n)}^p \leq C(n) \tilde{I}_1$$

where

$$(6.25) \quad \tilde{I}_1 := \sum_{L \in \mathcal{L}_E} \sum_{K \in T(L) \cap \mathcal{A}} \sum_{Q \in V_K \setminus L} \frac{|\tilde{f}_1(a_Q) - \tilde{f}_1(a_K)|^p}{(\text{diam } K)^{p-n}}.$$

We construct the family  $\mathcal{Q}$  and the mappings

$$\mathcal{Q} \ni Q \mapsto Q' \in \mathcal{K}_E \quad \text{and} \quad \mathcal{Q} \ni Q \mapsto Q'' \in \mathcal{K}_E$$

precisely as in Theorem 6.1, see formulas (6.7)-(6.10), but only for cubes  $K, Q$  from inequality (6.25), i.e., for contacting cubes.

As a result, we again obtain inequality (6.12). Note that, by Proposition 2.3, we have the following:

$$(6.26) \quad \sum_{Q \in \mathcal{Q}_1} \frac{(\text{diam } Q)^{n-p} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)}{\{(\text{diam } Q')^{n-p} + \mu(Q')\} \{(\text{diam } Q'')^{n-p} + \mu(Q'')\}} \leq C(n) \|f\|_{\Sigma}^p.$$

The crucial point of this construction is as follows: *the mappings*

$$\mathcal{Q} \ni Q \mapsto Q' \in \mathcal{K}_E \quad \text{and} \quad \mathcal{Q} \ni Q \mapsto Q'' \in \mathcal{K}_E$$

are “almost” one-to-one. Thus for every  $K \in \mathcal{K}_E$  there exist at most  $C(n)$  cubes  $Q \in \mathcal{Q}_1$  such that  $Q' = K$ . The same is true for the mapping  $Q \mapsto Q''$ .

This statement easily follows from a similar property of the “projection” operator  $\mathcal{PR} : \mathcal{L}_E \rightarrow E$  and inequality (6.24). In fact, let  $c_K$  be the center of  $K$ . Then, by part (ii) Proposition 6.7, there are at most  $C_1(n)$  lacunae  $L \in \mathcal{L}_E$  such that  $A_L = c_K$ . Each lacuna  $L$  from this family contains at most  $C_2(n)$  contacting cubes. For each such a cube, say  $H$ , there are at most  $C_3(n)$  cubes from other lacunae which contact with  $H$ . Finally, we obtain at most

$$\#\{Q \in \mathcal{Q}_1 : Q' = K\} \leq C_1(n) C_2(n) C_3(n).$$

The same estimate is true for  $Q''$ .

Let us estimate  $\|f_2\|_{L_p(\mathbf{R}^n; \mu)}$ . We again follow the prove of Theorem 6.1, see (6.14). We modify this inequality in the same fashion as we did this for the quantity  $I_1$ . We obtain

$$(6.27) \quad \int_{\mathbf{R}^n} |f_2|^p d\mu \leq C\{\tilde{I}_1 + I_2\}$$

where the quantities  $\tilde{I}_1$  and  $I_2$  are defined by (6.25) and (5.7) respectively.

We know that  $\tilde{I}_1$  can be estimated via the family  $\mathcal{Q}_1$  which we have constructed below. Let us estimate the quantity

$$(6.28) \quad I_2 = \sum_{K \in \mathcal{A}} \int_K |f - \tilde{f}_1(a_K)|^p d\mu + \sum_{K \in \mathcal{K}_E} \int_K |f - f_K|^p d\mu = I_3 + I_4.$$

First we estimate the quantity

$$I_4 := \sum_{K \in \mathcal{K}_E} \int_K |f - f_K|^p d\mu.$$

Let  $\mathcal{Q}_2 := \mathcal{K}_E$ . To each  $Q \in \mathcal{Q}_2$  we assign cubes  $Q'$  and  $Q''$  by letting  $Q' = Q'' = Q$ . Then

$$I_4 := \sum_{Q \in \mathcal{Q}_2} \int_{Q'} |f - f_{Q''}|^p d\mu \leq \sum_{Q \in \mathcal{Q}_2} \frac{1}{\mu(Q')} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y).$$

Since  $Q', Q'' \in \mathcal{K}_E$ , we have

$$\mu(Q') \sim (\text{diam } Q')^{n-p}, \quad \mu(Q'') \sim (\text{diam } Q'')^{n-p},$$

see Corollary 3.9. Hence

$$I_4 := \sum_{Q \in \mathcal{Q}_2} \int_{Q'} |f - f_{Q''}|^p d\mu \leq \sum_{Q \in \mathcal{Q}_2} \frac{1}{\mu(Q')} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)$$

so that

$$(6.29) \quad I_4 \leq C \sum_{Q \in \mathcal{Q}_2} \frac{(\text{diam } Q)^{n-p} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)}{\{(\text{diam } Q')^{n-p} + \mu(Q')\} \{(\text{diam } Q'')^{n-p} + \mu(Q'')\}}.$$

Clearly,  $Q', Q'' \subset \gamma Q$  for every  $Q \in \mathcal{Q}_2$  with  $\gamma = 1$ . Then, by Proposition 2.3, inequality (6.26) remains true after replacement the family  $\mathcal{Q}_1$  by  $\mathcal{Q}_2$ .

It is also clear that  $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \emptyset$ , and the (identical) mappings  $\mathcal{Q}_2 \ni Q \mapsto Q' \in \mathcal{K}_E$  and  $\mathcal{Q}_2 \ni Q \mapsto Q'' \in \mathcal{K}_E$  are one-to-one mappings.

We turn to the quantity

$$I_3 = \sum_{K \in \mathcal{A}} \int_K |f - \tilde{f}_1(a_K)|^p d\mu.$$

Since covering multiplicity  $M(W_E) \leq C(n)$ , we have

$$I_3 \leq C(n) \sum_{L \in \mathcal{L}_E} \int_{\tilde{U}_L} |f - \tilde{f}_1(A_L)|^p d\mu$$

where

$$\tilde{U}_L := \cup \{Q : Q \in L \cap \mathcal{A}\}.$$

We also recall that

$$\tilde{f}_1(A_L) = \frac{1}{\mu(K_L)} \int_{K_L} f d\mu$$

where  $K_L \in \mathcal{K}_E$  is the unique cube such that  $A_L = c_{K_L}$ . Hence

$$(6.30) \quad I_3 \leq C J_3$$

where

$$(6.31) \quad J_3 := \sum_{L \in \mathcal{L}_E} \frac{1}{\mu(K_L)} \iint_{\tilde{U}_L \times K_L} |f(x) - f(y)|^p d\mu(x) d\mu(y).$$

By definition of the family  $\mathcal{A}$ , see (4.17), for every cube  $Q \in \tilde{U}_L$  we have  $(\eta K_L) \cap Q = \emptyset$ . Hence

$$(6.32) \quad \text{dist}(Q, A_L) \geq \eta \text{diam } K_L.$$

On the other hand, by (6.23),  $A_L \subset \gamma Q$  so that

$$(6.33) \quad \text{dist}(Q, A_L) \leq \gamma \text{diam } Q$$

proving that

$$\text{diam } K_L \leq C \text{diam } Q.$$

These inequalities also imply that

$$(6.34) \quad K_L \subset \gamma_3 Q, \quad Q \in \tilde{U}_L.$$

Let us note that the family of sets

$$J := \{\tilde{U}_L \cup K_L : L \in \mathcal{L}_E\}$$

has covering multiplicity  $M(J) \leq C(n)$ .

To complete the proof of the theorem we need the following

**Lemma 6.12** *Let  $L \in \mathcal{L}_E$  and let  $f \in \Sigma$ . Then  $J_3 \leq C(n, p) \|f\|_\Sigma^p$ .*

*Proof.* Let  $f = f_1 + f_2$  where  $f_1 \in L_p^1(\mathbf{R}^n)$  and  $f_2 \in L_p(\mathbf{R}^n; \mu)$ . Then

$$\begin{aligned}
J(f; L) &:= \frac{1}{\mu(K_L)} \iint_{\tilde{U}_L \times K_L} |f(x) - f(y)|^p d\mu(x) d\mu(y) \\
&\leq 2^p \left\{ \frac{1}{\mu(K_L)} \iint_{\tilde{U}_L \times K_L} |f_1(x) - f_1(y)|^p d\mu(x) d\mu(y) \right. \\
&\quad \left. + \frac{1}{\mu(K_L)} \iint_{\tilde{U}_L \times K_L} |f_2(x) - f_2(y)|^p d\mu(x) d\mu(y) \right\} \\
&= 2^p \{I_1(L) + I_2(L)\}.
\end{aligned}$$

Let  $q := (p + n)/2$ . By (6.34),  $K_L \subset \gamma Q$  for every  $Q \in \tilde{U}_L$  so that  $Q \cup K_L \subset \gamma_1 Q$ . Then, by the Sobolev-Poincaré inequality, see Proposition 2.1, for every  $x, y \in Q \cup K_L$  we have

$$|f(x) - f(y)|^p \leq C(n, q) (\text{diam } Q)^p \left( \frac{1}{|\gamma_1 Q|} \int_{\gamma_1 Q} \|\nabla f(z)\|^q dz \right)^{\frac{p}{q}}.$$

Hence

$$\begin{aligned}
S_1(Q) &:= \frac{1}{\mu(K_L)} \iint_{Q \times K_L} |f_1(x) - f_1(y)|^p d\mu(x) d\mu(y) \\
&\leq C \mu(Q) (\text{diam } Q)^p \left( \frac{1}{|\gamma_1 Q|} \int_{\gamma_1 Q} \|\nabla f(z)\|^q dz \right)^{\frac{p}{q}}.
\end{aligned}$$

Recall that  $K_L \in \mathcal{K}_E$  so that, by Corollary 3.9,

$$\mu(K_L) \sim (\text{diam } K_L)^{n-p}.$$

Since  $Q \in W_E$ , by Corollary 4.3,

$$(6.35) \quad \mu(Q) \leq C (\text{diam } Q)^{n-p}.$$

Hence

$$\begin{aligned}
S_1(Q) &\leq C (\text{diam } Q)^{n-p} (\text{diam } Q)^p \left( \frac{1}{|\gamma_1 Q|} \int_{\gamma_1 Q} \|\nabla f(z)\|^q dz \right)^{\frac{p}{q}} \\
&\leq C |Q| \left( \frac{1}{|\gamma_1 Q|} \int_{\gamma_1 Q} \|\nabla f(z)\|^q dz \right)^{\frac{p}{q}}.
\end{aligned}$$

By (2.2),

$$S_1(Q) \leq C |Q| \left( \frac{1}{|\gamma_1 Q|} \int_{\gamma_1 Q} \|\nabla f(z)\|^q dz \right)^{\frac{p}{q}} \leq C \int_{\frac{1}{2}Q} \mathcal{M}[\|\nabla f_1(z)\|^q]^{\frac{p}{q}}(z) dz.$$

Hence

$$\begin{aligned} I_1(L) &:= \frac{1}{\mu(K_L)} \iint_{\tilde{U}_L \times K_L} |f_1(x) - f_1(y)|^p d\mu(x) d\mu(y) \\ &\leq \sum_{Q \in \tilde{U}_L} S_1(Q) \leq C \int_{U_L} \mathcal{M}[\|\nabla f_1(z)\|^q]^{\frac{p}{q}}(z) dz. \end{aligned}$$

Recall that  $U_L = \cup\{Q : Q \in L\}$ .

Since the sets  $\{U_L : L \in \mathcal{L}_E\}$  are pairwise disjoint, we obtain

$$A_1 := \sum_{L \in \mathcal{L}_E} I_1(L) \leq C \sum_{L \in \mathcal{L}_E} \int_{U_L} \mathcal{M}[\|\nabla f_1(z)\|^q]^{\frac{p}{q}}(z) dz \leq C \int_{\mathbf{R}^n} \mathcal{M}[\|\nabla f_1(z)\|^q]^{\frac{p}{q}}(z) dz.$$

Since  $p > q$ , by the Hardy-Littlewood maximal theorem,

$$A_1 \leq C \int_{\mathbf{R}^n} (\|\nabla f_1(z)\|^q)^{\frac{p}{q}}(z) dz = C \int_{\mathbf{R}^n} \|\nabla f_1(z)\|^p(z) dz = \|f_1\|_{L_p^1(\mathbf{R}^n)}^p.$$

Let us estimate the quantity

$$I_2(L) := \frac{1}{\mu(K_L)} \iint_{\tilde{U}_L \times K_L} |f_2(x) - f_2(y)|^p d\mu(x) d\mu(y).$$

We have

$$I_2(L) \leq \frac{2^p}{\mu(K_L)} \left\{ \iint_{\tilde{U}_L \times K_L} |f_2(x)|^p d\mu(x) d\mu(y) + \iint_{\tilde{U}_L \times K_L} |f_2(y)|^p d\mu(x) d\mu(y) \right\}.$$

Hence

$$I_2(L) \leq 2^p \left\{ \int_{\tilde{U}_L} |f_2(x)|^p d\mu(x) + \frac{\mu(\tilde{U}_L)}{\mu(K_L)} \int_{K_L} |f_2(y)|^p d\mu(y) \right\}.$$

Prove that

$$(6.36) \quad \mu(\tilde{U}_L) \leq C \mu(K_L).$$

By (6.35),

$$\mu(\tilde{U}_L) \leq \sum_{Q \in L \cap \mathcal{A}} \mu(Q) \leq C \sum_{Q \in L \cap \mathcal{A}} (\text{diam } Q)^{n-p}.$$

Note that for every  $Q \in L \cap \mathcal{A}$  and every  $x \in Q$  we have

$$(6.37) \quad \|x - A_L\| \sim \text{diam } Q.$$

In fact, since  $Q \in W_E$ , its diameter  $\text{diam } Q \sim \text{dist}(Q, E)$  so that  $\text{diam } Q \leq C \text{dist}(Q, E)$ . Hence

$$\text{diam } Q \leq C \|x - A_L\|.$$

On the other hand, by (6.33),

$$\|x - A_L\| \leq \text{diam } Q + \text{dist}(Q, A_L) \leq C \text{diam } Q$$

proving (6.37).

Hence

$$\mu(\tilde{U}_L) \leq C \sum_{Q \in L \cap \mathcal{A}} (\text{diam } Q)^{n-p} \leq C \sum_{Q \in L \cap \mathcal{A}} \int_Q \frac{dx}{\|x - A_L\|^p}.$$

Note that, by (6.32), we have

$$\|x - A_L\| \geq \eta \text{diam } K_L, \quad x \in \tilde{U}_L.$$

We also note that covering multiplicity  $M(W_E) \leq N(n)$ . Hence

$$\mu(\tilde{U}_L) \leq CN(n) \int_{\|x - A_L\| \geq \eta \text{diam } K_L} \frac{dx}{\|x - A_L\|^p}.$$

We obtain

$$\mu(\tilde{U}_L) \leq C(\text{diam } K_L)^{n-p} \sim \mu(K_L)$$

proving (6.36).

Hence

$$I_2(L) \leq C \left\{ \int_{\tilde{U}_L} |f_2(x)|^p d\mu(x) + \int_{K_L} |f_2(y)|^p d\mu(x) \right\}.$$

Recall that the mapping  $L \mapsto K_L$  is an “almost” one-to-one, so that covering multiplicity of the family  $\{K_L : L \in \mathcal{L}_E\}$  is bounded by a constant  $N(n)$ . Since the sets  $\{\tilde{U}_L : L \in \mathcal{L}_E\}$  are pairwise disjoint, we obtain

$$\begin{aligned} A_2 &:= \sum_{L \in \mathcal{L}_E} I_2(L) \leq C \sum_{L \in \mathcal{L}_E} \int_{\tilde{U}_L} |f_2(x)|^p d\mu(x) + C \sum_{L \in \mathcal{L}_E} \int_{K_L} |f_2(y)|^p d\mu(x) \\ &\leq C \int_{\mathbf{R}^n} |f_2(x)|^p d\mu(x) + C N(n) \int_{\mathbf{R}^n} |f_2(x)|^p d\mu(x) \\ &\leq C \int_{\mathbf{R}^n} |f_2(x)|^p d\mu(x) \end{aligned}$$

proving that

$$A_2 \leq \|f_2\|_{L_p(\mathbf{R}^n; \mu)}^p.$$

Finally,

$$\begin{aligned} J_3 &= \sum_{L \in \mathcal{L}_E} \frac{1}{\mu(K_L)} \iint_{\tilde{U}_L \times K_L} |f(x) - f(y)|^p d\mu(x) d\mu(y) \\ &\leq C(A_1 + A_2) \leq C(\|f_1\|_{L_p^1(\mathbf{R}^n)} + \|f_2\|_{L_p(\mathbf{R}^n; \mu)})^p. \end{aligned}$$

taking the infimum in this inequality over all functions  $f_1 \in L_p^1(\mathbf{R}^n)$  and  $f_2 \in L_p(\mathbf{R}^n; \mu)$  such that  $f = f_1 + f_2$  we obtain the statement of the lemma.  $\blacksquare$

Let us finish the proof of the theorem.

Let us enumerate the cubes of the family  $\mathcal{Q}_1$ :

$$\mathcal{Q}_1 = \{Q_i : i \in \mathbf{N}\}.$$

Let

$$D_1(f) := \sum_{i=1}^{\infty} \alpha_i \iint_{Q'_i \times Q''_i} |f(x) - f(y)|^p d\mu(x) d\mu(y)$$

where

$$\alpha_i := \frac{(\text{diam } Q_i)^{n-p}}{\{(\text{diam } Q'_i)^{n-p} + \mu(Q'_i)\} \{(\text{diam } Q''_i)^{n-p} + \mu(Q''_i)\}}$$

Then, by (6.12),

$$\|f_1\|_{L_p^1(\mathbf{R}^n)}^p \leq C D_1(f).$$

Recall that we have constructed the families of cubes

$$\mathcal{G}_1 := \{Q'_i : i \in \mathbf{N}\} \quad \text{and} \quad \mathcal{H}_1 := \{Q''_i : i \in \mathbf{N}\}$$

in such a way that  $M(\mathcal{G}_1), M(\mathcal{H}_1) \leq C(n)$ . We have also proved that

$$D_1(f) \leq C \|f\|_{\Sigma}^p.$$

We have also defined a family  $\mathcal{Q}_2$  of cubes

$$\mathcal{Q}_2 = \{K_i : i \in \mathbf{N}\}.$$

Let

$$D_2(f) := \sum_{i=1}^{\infty} \beta_i \iint_{K'_i \times K''_i} |f(x) - f(y)|^p d\mu(x) d\mu(y)$$

where

$$\beta_i := \frac{(\text{diam } K_i)^{n-p}}{\{(\text{diam } K'_i)^{n-p} + \mu(K'_i)\} \{(\text{diam } K''_i)^{n-p} + \mu(K''_i)\}}.$$

By (6.29),

$$(6.38) \quad I_4 \leq C D_2(f).$$

We know that

$$D_2(f) \leq C \|f\|_{\Sigma}^p.$$



Furthermore, we know that for the families of cubes

$$\mathcal{G}_2 := \{K'_i : i \in \mathbf{N}\} \quad \text{and} \quad \mathcal{H}_2 := \{K''_i : i \in \mathbf{N}\}$$

we have  $M(\mathcal{G}_2), M(\mathcal{H}_2) \leq C(n)$ .

Finally, we have constructed two families of sets

$$\mathcal{G}_3 := \{\tilde{U}_L : L \in \mathcal{L}_E\} \quad \text{and} \quad \mathcal{H}_3 := \{K_L : L \in \mathcal{L}_E\}$$

with certain properties. Let us enumerate the sets of these families with preservation of the correspondence  $\tilde{U}_L \leftrightarrow K_L$ :

$$\mathcal{G}_3 := \{S_i : i \in \mathbf{N}\} \quad \text{and} \quad \mathcal{H}_3 := \{T_i : i \in \mathbf{N}\}.$$

We know that  $M(\mathcal{G}_3), M(\mathcal{H}_3) \leq C(n)$ . Note that the quantity  $J_3$  defined by (6.31) can be written in the form  $J_3 = D_3(f)$  where

$$D_3(f) := \sum_{i=1}^{\infty} \theta_i \iint_{S_i \times T_i} |f(x) - f(y)|^p d\mu(x) d\mu(y)$$

where

$$\theta_i := 1/\mu(T_i).$$

Furthermore, by Lemma 6.12,

$$J_3 = D_3(f) \leq C\|f\|_{\Sigma}.$$

Combining inequalities (6.27), (6.28), (6.30) and (6.38) with definition of  $J_3$ , see (6.31), we obtain

$$\|f_2\|_{L_p(\mathbf{R}^n; \mu)}^p \leq C\{D_1(f) + D_2(f) + D_3(f)\}.$$

Hence,

$$\|f\|_{\Sigma}^p \leq 2^p \{\|f_1\|_{L_p^1(\mathbf{R}^n)}^p + \|f_2\|_{L_p(\mathbf{R}^n; \mu)}^p\}^p \leq C\{D_1(f) + D_2(f) + D_3(f)\}$$

so that

$$\|f\|_{\Sigma}^p \sim D_1(f) + D_2(f) + D_3(f).$$

Theorem 6.11 is completely proved. ■

## 7. Further results and comments.

**7.1. Modifications of Theorem 1.3.** In this subsection we present several versions of the criterion for calculation of the norm of a function in the space  $L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$ . First of them is Theorem 1.6 which we have formulated in Section 1. Its proof is very short.

*Proof of Theorem 1.6.* The necessity part of the theorem directly follows from the necessity part of Theorem 1.3. In fact, the left hand side of inequality (1.4) is majorized (up to an absolute constant) by the left hand side of (1.2) provided inequality (1.3) holds. In turn, the sufficiency part of the theorem immediately follows from Theorem 3.1. ■

**Remark 7.1** Inequality (1.3) of Theorem 1.6 can be replaced by weaker conditions

$$(7.1) \quad \mu(Q') \leq (\text{diam } Q')^{n-p}, \quad \mu(Q'') \leq (\text{diam } Q'')^{n-p} \leq C \mu(Q''),$$

and

$$(7.2) \quad \mu(Q') \leq C \mu(Q'')$$

where  $C = C(n, p)$  is a constant depending only on  $n$  and  $p$ .

Note that (7.2) is equivalent to the inequality

$$\text{diam } Q'' \leq C \text{diam } Q'$$

provided the inequalities in (7.1) hold.

In fact, in all our definitions of the cube  $Q''$  (see Propositions 4.11, 4.15 and 5.5) we have  $Q'' \in \mathcal{K}_E$  so that  $\mu(Q'') \sim (\text{diam } Q'')^{n-p}$ , see (3.28).

On the other hand, in Propositions 4.11 and 4.15 both  $Q'$  and  $Q''$  belong to  $\mathcal{K}_E$  so that in this case without loss of generality we may assume that (7.2) holds with  $C = 1$ . In the same way we can define the cubes  $Q'$  and  $Q''$  in Propositions 5.5 whenever  $Q = Q' = Q'' \in \mathcal{K}_E$ , see (5.10).

In the remaining case, see (5.9),

$$Q' = K \in \mathcal{A} \quad \text{and} \quad Q'' = K^{(a_K)} \in \mathcal{K}_E.$$

Then, by Lemma 4.9,  $Q'' \subset (22\tau^2)Q'$  so that  $\text{diam } Q'' \leq C \text{diam } Q'$ . But, by (3.28) and (4.2),

$$\mu(Q'') \sim (\text{diam } Q'')^{n-p} \quad \text{and} \quad \mu(Q') \leq C(\text{diam } Q')^{n-p}$$

proving (7.2).  $\triangleleft$

**Remark 7.2** Let us replace inequality (1.2) in Theorem 1.3 with the following one:

$$(7.3) \quad \sum_{Q \in \mathcal{Q}} \frac{\frac{1}{\mu(Q')\mu(Q'')} \iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)}{(\text{diam } Q)^{p-n} \{1 + (\text{diam } Q')^{n-p}/\mu(Q') + (\text{diam } Q'')^{n-p}/\mu(Q'')\}} \leq \lambda$$

Then the result of Theorem 1.3 remains true after such a modification. Thus we obtain another criterion for calculation of the norm in the space  $L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$ .

In fact, the necessity part of this new criterion follows from Proposition 2.3 (with  $\mathcal{S} = \mathcal{Q}$ ,  $S_{Q'} = Q'$  and  $S_{Q''} = Q''$ ). In turn, the sufficiency directly follows from the sufficiency part of Theorem 1.3 because the left hand side of inequality (1.2) does not exceed the left hand side of (7.3). See definition (2.10) and inequality (2.11).  $\triangleleft$

The criterion (7.3) and previous results lead us to the following result formulated in the spirit of Theorem 1.6.

**Theorem 7.3** *Let us replace inequality (1.4) in the formulation of Theorem 1.6 by the inequality*

$$(7.4) \quad \sum_{Q \in \mathcal{Q}} \left( \frac{\text{diam } Q' \text{ diam } Q''}{\text{diam } Q} \right)^{p-n} \frac{\iint_{Q' \times Q''} |f(x) - f(y)|^p d\mu(x) d\mu(y)}{(\text{diam } Q')^{p-n} \mu(Q') + (\text{diam } Q'')^{p-n} \mu(Q'')} \leq \lambda.$$

*Then, after such a modification the result of Theorem 1.6 remains true.*

*Proof.* Clearly, by (1.3), the left hand side of (1.4) is smaller than the left hand side of (7.4) so that the sufficiency follows from the sufficiency part of Theorem 1.6. On the other hand the left hand side of (7.4) is smaller (up to an absolute constant) than to the left hand side of (7.3) provided inequality (1.3) is satisfied. But as we have seen in Remark 7.2, the necessity of (7.3) follows from Proposition 2.3. This proves the necessity part of Theorem 7.3.  $\blacksquare$

**Remark 7.4** In all modifications of the main result we may assume that the cubes  $Q', Q''$  belong to a certain family  $\tilde{\mathcal{Q}}$  of pairwise disjoint cubes which may be different from the family  $\mathcal{Q}$ . For instance, Theorem 1.6 can be modified in the following way:

**Theorem 7.5** *A function  $f \in L_{p,loc}(\mathbf{R}^n; \mu)$  belongs to the space  $L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$ ,  $n < p < \infty$ , if and only if there exists a constant  $\lambda > 0$  which satisfies the following conditions for a certain absolute constant  $\gamma$ : Let  $\mathcal{Q}$  and  $\tilde{\mathcal{Q}}$  be arbitrary finite families of pairwise disjoint cubes in  $\mathbf{R}^n$ . Suppose that to each cube  $Q \in \mathcal{Q}$  we have arbitrarily assigned two cubes  $Q', Q'' \in \tilde{\mathcal{Q}}$  such that  $Q' \cup Q'' \subset \gamma Q$  and inequality (1.3) is satisfied.*

*Then inequality (1.4) holds. Furthermore,  $\|f\|_{\Sigma} \sim \inf \lambda^{\frac{1}{p}}$  with constants of equivalence depending only on  $n$  and  $p$ .*

*(Necessity).* We apply Proposition 2.3 to  $\mathcal{S} = \tilde{\mathcal{Q}}$ ,  $S_{Q'} = Q'$  and  $S_{Q''} = Q''$  and prove that inequality (7.3) holds. As we have noted in Remark 7.2, the left hand side of inequality (1.2) does not exceed the left hand side of (7.3) which proves the necessity.

*(Sufficiency).* The sufficiency follows from the sufficiency part of Theorem 1.3 which is proven for the case  $\tilde{\mathcal{Q}} = \mathcal{Q}$ .  $\triangleleft$

**Remark 7.6** As we have mentioned in Section 1, Theorem 1.3 and its variants have important and interesting applications to the Whitney-type problems of characterizations of restrictions of Sobolev functions to subsets of  $\mathbf{R}^n$ . In particular, in [12] we need a variant of Theorem 1.6 formulated in terms of families of Euclidean balls rather than cubes.

**Theorem 7.7** *Let  $n < p < \infty$  and let  $\mu$  be a non-trivial non-negative Borel measure on  $\mathbf{R}^n$ . A function  $f \in L_{p,loc}(\mathbf{R}^n; \mu)$  belongs to the space  $L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$  if and only if there exists a constant  $\lambda > 0$  which satisfy all of the following conditions for a certain absolute positive constant  $\gamma$ : Let  $\mathcal{B}$  be an arbitrary finite family of pairwise disjoint balls in  $\mathbf{R}^n$ . Suppose that to each ball  $B \in \mathcal{B}$  we have arbitrarily assigned two balls  $B', B'' \in \mathcal{B}$  such that  $B' \cup B'' \subset \gamma B$  and*

$$(\text{diam } B')^{p-n} \mu(B') + (\text{diam } B'')^{p-n} \mu(B'') \leq 1.$$

Then the following inequality

$$\sum_{B \in \mathcal{Q}} \left( \frac{\text{diam } B' \text{ diam } B''}{\text{diam } B} \right)^{p-n} \iint_{B' \times B''} |f(x) - f(y)|^p d\mu(x) d\mu(y) \leq \lambda$$

holds.

Furthermore,  $\|f\|_{\Sigma} \sim \inf \lambda^{\frac{1}{p}}$  with constants of equivalence depending only on  $n$  and  $p$ .

*A sketch of the proof.* The proof follows precisely the scheme of the proof of Theorem 1.6.

There is only one place in this scheme where we have to slightly change formulations of corresponding results. We mean an analogue of the Whitney covering Theorem 4.1 for Euclidean balls. Of course, in this case we can not cover the open set  $\mathbf{R}^n \setminus E$  by *non-overlapping* balls  $B$  such that  $\text{diam } B \sim \text{dist}(B, E)$ . Nevertheless for our purpose it suffice to cover  $\mathbf{R}^n \setminus E$  by a family  $\widetilde{W}_E$  of balls whose *covering multiplicity* is bounded by a constant  $N = N(n)$  depending only on  $n$ . In other words, every point  $x \in \mathbf{R}^n$  is covered at most  $N$  balls from the family  $\widetilde{W}_E$ .

The existence of a Whitney-type covering of such a kind follows from a general result proven by M. Guzman [6]. (Note that this result relies on the Besicovitch covering theorem [3].)  $\triangleleft$

## 7.2. The $K$ -functional for the couple $\vec{A} = (L_p(\mathbf{R}^n; \mu), L_p^1(\mathbf{R}^n))$ .

Theorem 1.3 and its modifications presented in the previous subsection enable us to give various explicit formulas for the  $K$ -functional of the Banach couple

$$\vec{A} = (L_p(\mathbf{R}^n; \mu), L_p^1(\mathbf{R}^n)).$$

We recall that, for each  $t > 0$

$$K(t; f : \vec{A}) := \inf \{ \|f_1\|_{L_p(\mathbf{R}^n; \mu)} + t \|f_2\|_{L_p^1(\mathbf{R}^n)} : f_1 + f_2 = f, f_1 \in L_p(\mathbf{R}^n; \mu), f_2 \in L_p^1(\mathbf{R}^n) \}$$

so that  $\|f\|_{\Sigma} = K(1; f : \vec{A})$  and

$$(7.5) \quad K(t; f : \vec{A}) = t \|f\|_{\Sigma_t}$$

where

$$(7.6) \quad \Sigma_t := L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \tfrac{1}{t^p} \mu).$$

See Remark 1.4.

In Section 1 we have presented such a formula for  $K(\cdot; f : \vec{A})$ . This result directly follows from Theorem 7.3 and equalities (7.5) and (7.6).

Let us prove that the  $K$ -functional of the couple  $\vec{A}$  can be *quasi-linearized*, i.e., for each  $t > 0$  there exist continuous linear operators

$$T_1[t] : \Sigma \rightarrow L_p^1(\mathbf{R}^n) \quad \text{and} \quad T_2[t] : \Sigma \rightarrow L_p(\mathbf{R}^n; \mu)$$

such that

$$T_1[t] + T_2[t] = Id_{\Sigma}$$

and for every  $f \in \sum$  the following inequality

$$\|T_2[t](f)\|_{L_p(\mathbf{R}^n; \mu)} + t\|T_1[t](f)\|_{L_p^1(\mathbf{R}^n)} \leq C(n, p)K(t; f : \vec{A}).$$

holds. Here as before  $\sum = L_p^1(\mathbf{R}^n) + L_p(\mathbf{R}^n; \mu)$ .

This property easily follows from Theorem 1.5 and equalities (7.5) and (7.6). In fact, let us apply Theorem 1.5 to the measure  $\mu_t := \mu/t^p$ . By this theorem there exist continuous linear operators

$$T_1[t] : \sum_t \rightarrow L_p^1(\mathbf{R}^n) \quad \text{and} \quad T_2[t] : \sum_t \rightarrow L_p(\mathbf{R}^n; \mu_t)$$

such that

$$T_1[t] + T_2[t] = Id_{\sum_t}$$

and

$$(7.7) \quad \|T_1[t]\|_{\sum_t \rightarrow L_p^1(\mathbf{R}^n)} + \|T_2[t]\|_{\sum_t \rightarrow L_p(\mathbf{R}^n; \mu)} \leq C(n, p).$$

Since

$$\|f\|_{L_p(\mathbf{R}^n; \mu_t)} = \frac{1}{t} \|f\|_{L_p(\mathbf{R}^n; \mu)},$$

the Banach space  $L_p(\mathbf{R}^n; \mu_t)$  coincides with the Banach space  $L_p(\mathbf{R}^n; \mu)$  proving that the Banach spaces  $\sum_t$  and  $\Sigma$  coincide as well. Hence

$$T_1[t] : \sum \rightarrow L_p^1(\mathbf{R}^n) \quad \text{and} \quad T_2[t] : \sum \rightarrow L_p(\mathbf{R}^n; \mu)$$

and

$$T_1[t] + T_2[t] = Id_{\Sigma}.$$

Furthermore, by (7.5) and (7.7), for every  $f \in \sum$  we have

$$\|T_1[t](f)\|_{L_p^1(\mathbf{R}^n)} \leq C\|f\|_{\sum_t} = CK(t; f : \vec{A})/t,$$

and

$$\|T_2[t](f)\|_{L_p(\mathbf{R}^n; \mu_t)} = \|T_2[t](f)\|_{L_p(\mathbf{R}^n; \mu)}/t \leq C\|f\|_{\sum_t} = CK(t; f : \vec{A})/t.$$

We obtain

$$t\|T_1[t](f)\|_{L_p^1(\mathbf{R}^n)} \leq CK(t; f : \vec{A}),$$

and

$$\|T_2[t](f)\|_{L_p(\mathbf{R}^n; \mu)} \leq CK(t; f : \vec{A}).$$

Hence

$$\|T_2[t](f)\|_{L_p(\mathbf{R}^n; \mu)} + t\|T_1[t](f)\|_{L_p^1(\mathbf{R}^n)} \leq C(n, p)K(t; f : \vec{A})$$

proving that the  $K$ -functional of the couple  $\vec{A} = (L_p(\mathbf{R}^n; \mu), L_p^1(\mathbf{R}^n))$  is quasi-linearizable.

Finally, we remark that Z. Ditzian and V. Totik [4] have studied a number of variants of the  $K$ -functional for the Banach couple  $\vec{B} = (L_p(\mathbf{R}), L_p^1(\mathbf{R}; \mu))$  where  $L_p^1(\mathbf{R}; \mu)$  is a homogeneous Sobolev space on  $\mathbf{R}$  with respect to the measure  $\mu$ . This space is defined by the finiteness of the seminorm

$$\|f\|_{L_p^1(\mathbf{R}; \mu)} := \left( \int_{\mathbf{R}} |f'(x)|^p d\mu(x) \right)^{\frac{1}{p}}.$$

At first sight, the couples  $\vec{A} = (L_p(\mathbf{R}; \mu), L_p^1(\mathbf{R}))$  and  $\vec{B} = (L_p(\mathbf{R}), L_p^1(\mathbf{R}; \mu))$  look very similar to each other. Nevertheless, in general, the  $K$ -functionals of these couples are very different from each other, and their calculations require different methods and ideas.

**7.3. Theorem 3.3 and subfamilies of “minimal” pairwise disjoint cubes.** The result of Theorem 3.3 can be reformulated in a purely geometrical way. In fact, given a function  $w : \mathbf{R}^n \rightarrow (0, \infty)$  consider a family of cubes

$$\mathcal{A}_w = \{Q = Q(x, w(x)) : x \in \mathbf{R}^n\}$$

and a subfamily of  $\mathcal{A}_w$

$$\mathcal{B} = \{K = Q(x, w(x)) : x \in S\}$$

where  $S$  is the set determined in Theorem 3.3. Then the conditions (i) and (ii) from this proposition are equivalent to the following statements:

- (i'). For every cube  $Q \in \mathcal{A}_w$  there exists a cube  $K \in \mathcal{B}$  such that  $K \subset 83Q$ ;
- (ii'). The cubes of the family  $\mathcal{B}$  are pairwise disjoint.

Thus Theorem 3.3 states that for every function  $w$  satisfying condition (3.4) the family  $\mathcal{A}_w$  contains a subfamily  $\mathcal{B}$  satisfying conditions (i') and (ii').

This geometrical reformulation of the proposition motivates the following

**Question 7.8** *Let  $\mathcal{A}$  be a family of cubes in  $\mathbf{R}^n$ . Under what conditions on  $\mathcal{A}$  there exists a subfamily  $\mathcal{B}$  of  $\mathcal{A}$  such that:*

- (a). *For every cube  $Q \in \mathcal{A}$  there exists a cube  $K \in \mathcal{B}$  such that  $K \cap Q \neq \emptyset$  and  $\text{diam } K \leq \text{diam } Q$ ;*
- (b). *The cubes of the family  $\mathcal{B}$  are pairwise disjoint?*

**Remark 7.9** Note that the condition (a) implies the inclusion  $2Q \supset K$ .

Also, let  $\gamma \geq 1$  and let  $\gamma\mathcal{A} = \{\gamma Q : Q \in \mathcal{A}\}$ . Then the existence of a collection  $\mathcal{B}$  satisfying the conditions (a) and (b) for the family  $\gamma\mathcal{A}$  implies the existence of a subfamily  $\tilde{\mathcal{B}}$  of the family  $\mathcal{A}$  such that: (a'). For each  $Q \in \mathcal{A}$  there exists  $K \in \tilde{\mathcal{B}}$  such that  $K \subset (2\gamma)Q$ ; (b'). the cubes  $\{\gamma K : K \in \tilde{\mathcal{B}}\}$  are pairwise disjoint.

Clearly, one can put  $\tilde{\mathcal{B}} = \frac{1}{\gamma}\mathcal{B}$ .  $\triangleleft$

V. Dolnikov kindly drew the author's attention to the fact that a family  $\mathcal{B}$  satisfying conditions (a) and (b) exists whenever  $\mathcal{A}$  is an arbitrary *finite* collection of cubes. Here is a short Dolnikov's proof of this statement.

Let  $K_1$  be a cube of the minimal diameter among all the cubes of the family  $\mathcal{A}_1 := \mathcal{A}$ . By  $G_1$  we denote all cubes of  $\mathcal{A}_1$  which intersect  $K_1$ .

We put  $\mathcal{A}_2 := \mathcal{A}_1 \setminus G_1$ . If  $\mathcal{A}_2 = \emptyset$  we stop and put  $\mathcal{B} = \{K_1\}$ . If  $\mathcal{A}_2 \neq \emptyset$ , by  $K_2$  we denote a cube of the minimal diameter among all the cubes of the family  $\mathcal{A}_2$ . We continue this procedure. Since  $\mathcal{A}$  is finite, this process will stop on a certain (finite) step  $m$ .

As a result we obtain a finite collection of pairwise disjoint cubes  $\mathcal{B} = \{K_1, \dots, K_m\}$  and a partition  $\{G_1, \dots, G_m\}$  of  $\mathcal{A}$  such that for each  $1 \leq i \leq m$  the following conditions are satisfied: the cube  $K_i \in G_i$ ,  $K_i \cap Q \neq \emptyset$ , and  $\text{diam } K_i \leq \text{diam } Q$  for every  $Q \in G_i$ . Clearly, the collection  $\mathcal{B}$  satisfies the conditions (a) and (b) of Question 7.8.

Let us also note that for an *infinite* collection  $\mathcal{A}$  of cubes in  $\mathbf{R}^n$  a family of cubes  $\mathcal{B}$  satisfying conditions (a) and (b) of Question 7.8 in general does not exist. For instance,  $\mathcal{B}$  does not exist whenever  $\mathcal{A} = \{Q(0, \frac{1}{n}) : n = 1, 2, \dots\}$ .

These examples show that a certain “continuity” condition (apparently in the spirit of the condition (3.4)) should be posed on the collection  $\mathcal{A}$  to provide the existence of a subfamily  $\mathcal{B}$  satisfying conditions (a) and (b) of Question 7.8.

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